Anisotropic Diffusion in SPH

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Diffusion in SPH

$$\frac{\partial T}{\partial t} = \nabla \cdot (\mathbf{K} \nabla T)$$

$$\frac{\partial^2 T}{\partial r^i \partial r^j} \approx \sum_b \frac{m_b}{\rho_b} T(r_b)(5\hat{r}^i \hat{r}^j - \delta_{ij}) F_{ab}$$

Heat Conduction

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho} \nabla \cdot (k \nabla T_{\mathrm{m}})$$

Radiative Transfer

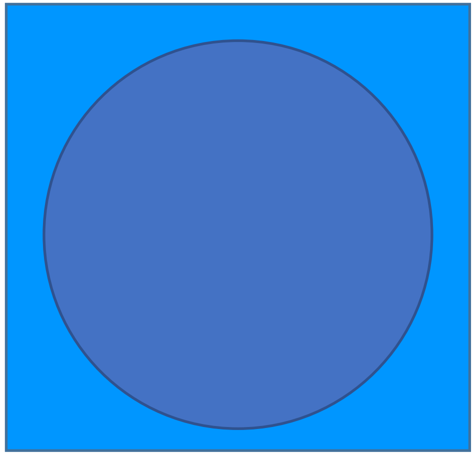
$$-\frac{\nabla \cdot \boldsymbol{F}}{\rho} = \frac{1}{\rho} \nabla \cdot \left(\frac{c\lambda}{\kappa\rho} \nabla E \right)$$

Gas-Dust Fraction

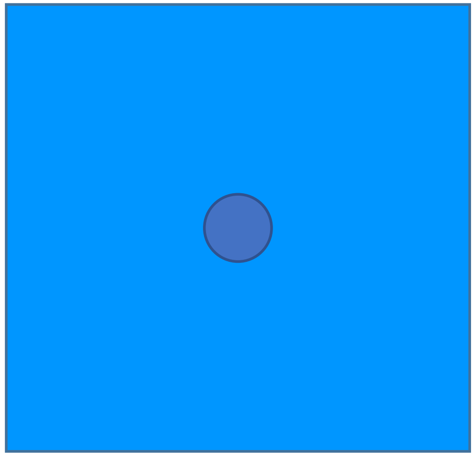
$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{1}{\rho} \nabla \cdot (\epsilon t_{\mathrm{s}} \nabla P)$$

Isotropic

$$\boldsymbol{K} = k * \begin{array}{cc} \begin{matrix} x & y \end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} x \\ y \end{matrix} \end{array}$$

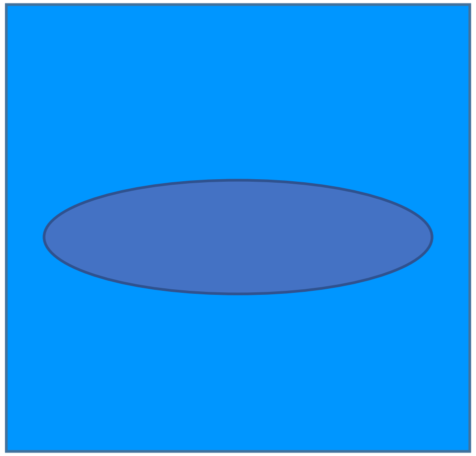


Initial



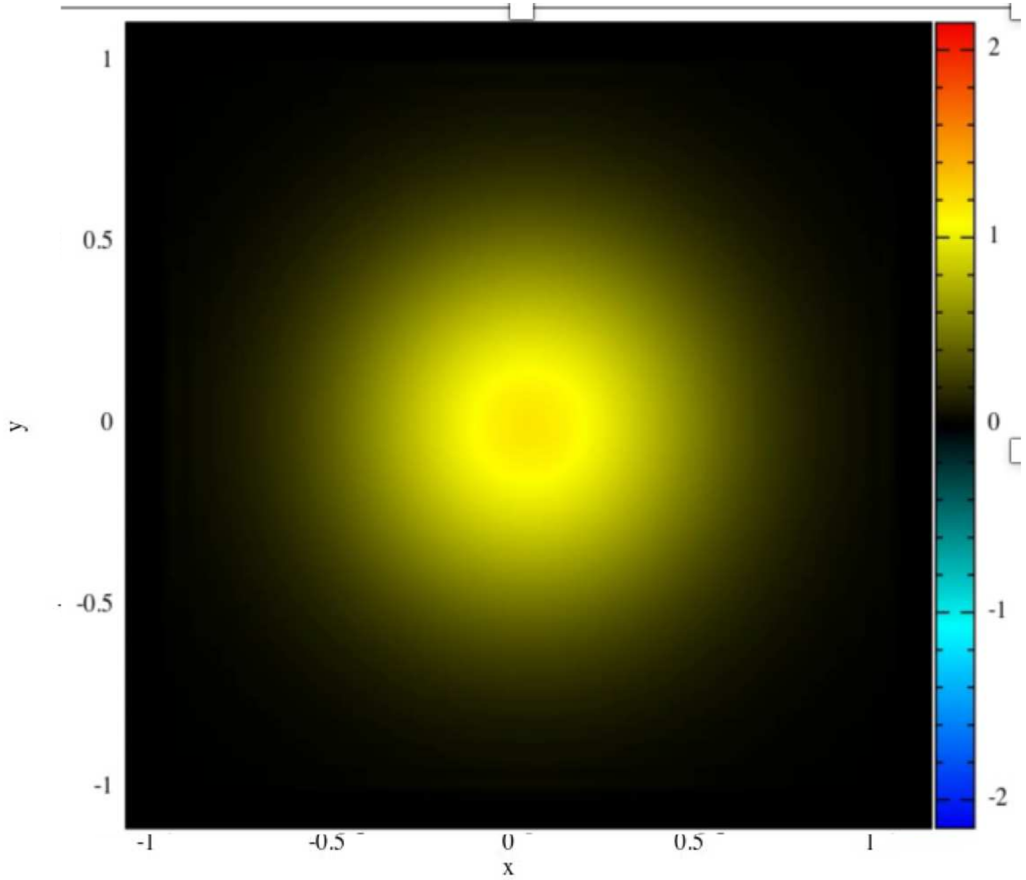
Anisotropic

$$\boldsymbol{K} = \begin{array}{cc} \begin{matrix} x & y \end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{matrix} x \\ y \end{matrix} \end{array}$$

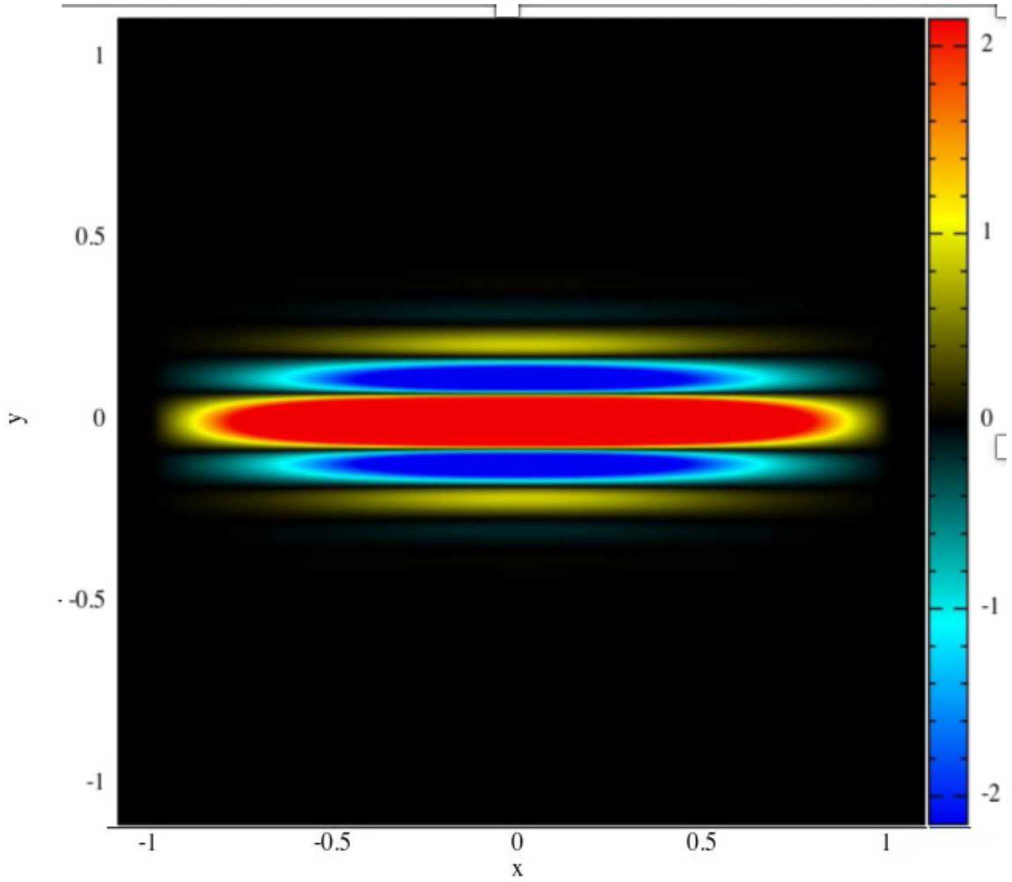


Question

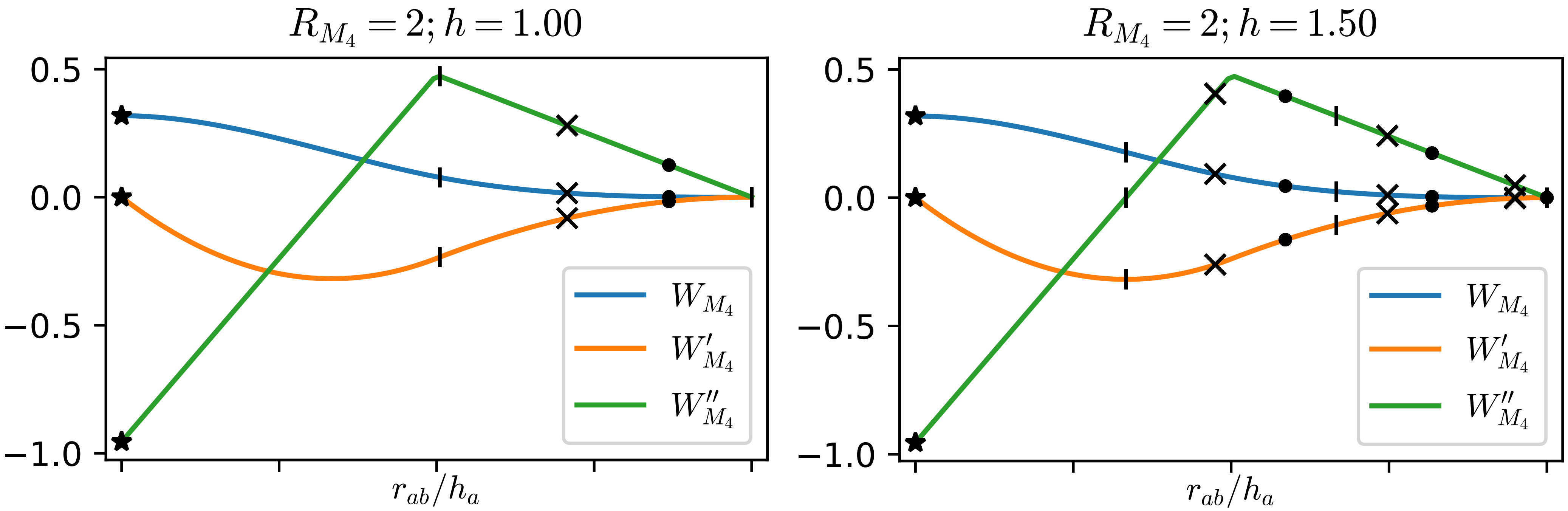
Isotropic



Anisotropic



Smoothing Length



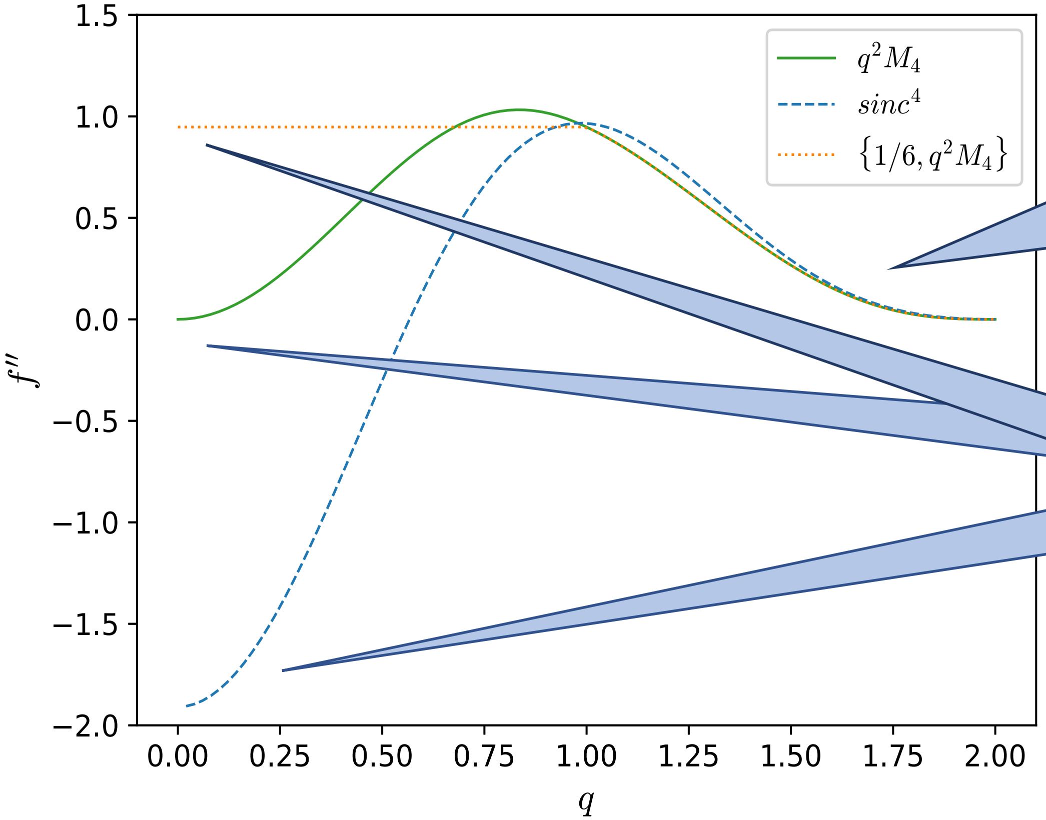
Particles over the shape of M_4 − spline for different smoothing length.

| − r_{ab} have only one non-zero component

X − r_{ab} have two non-zero component

○ − all of the r_{ab} components are not zero

Good Outer Part – Good Kernel

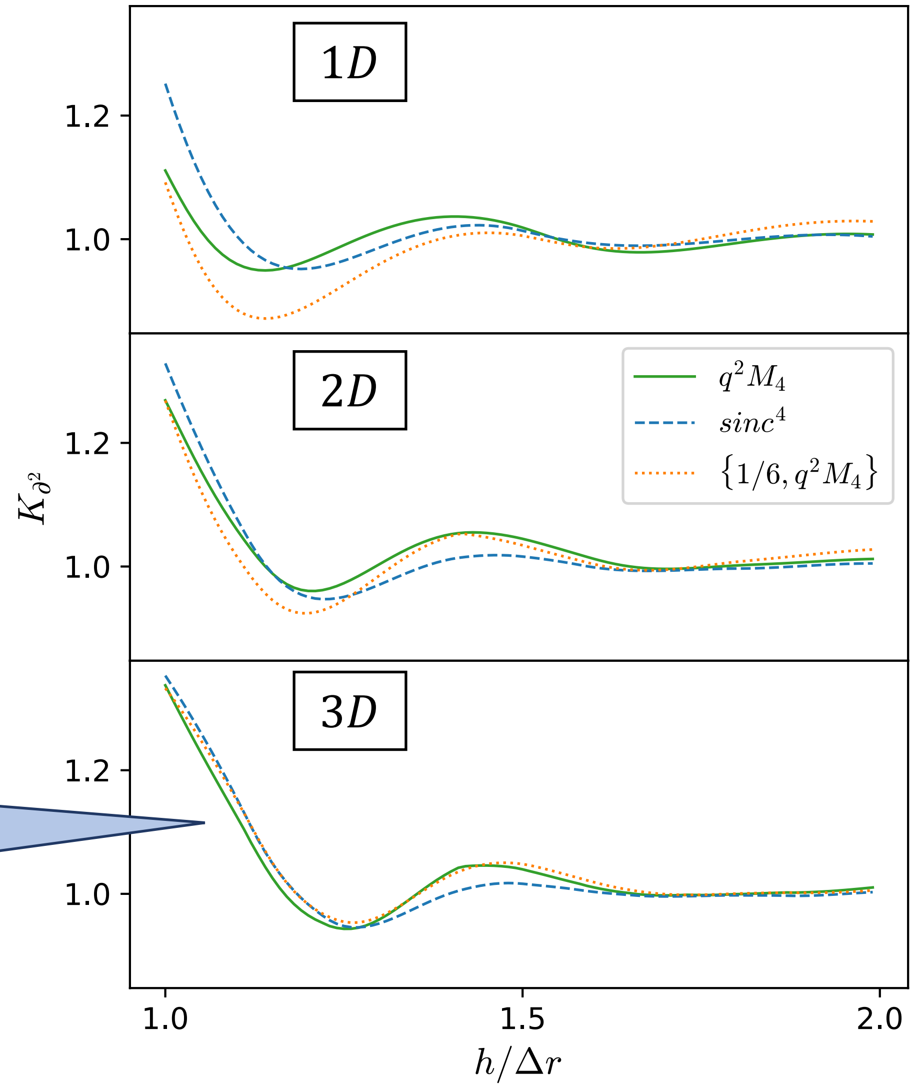


Second derivative shapes of kernel functions.

Outer part more important!

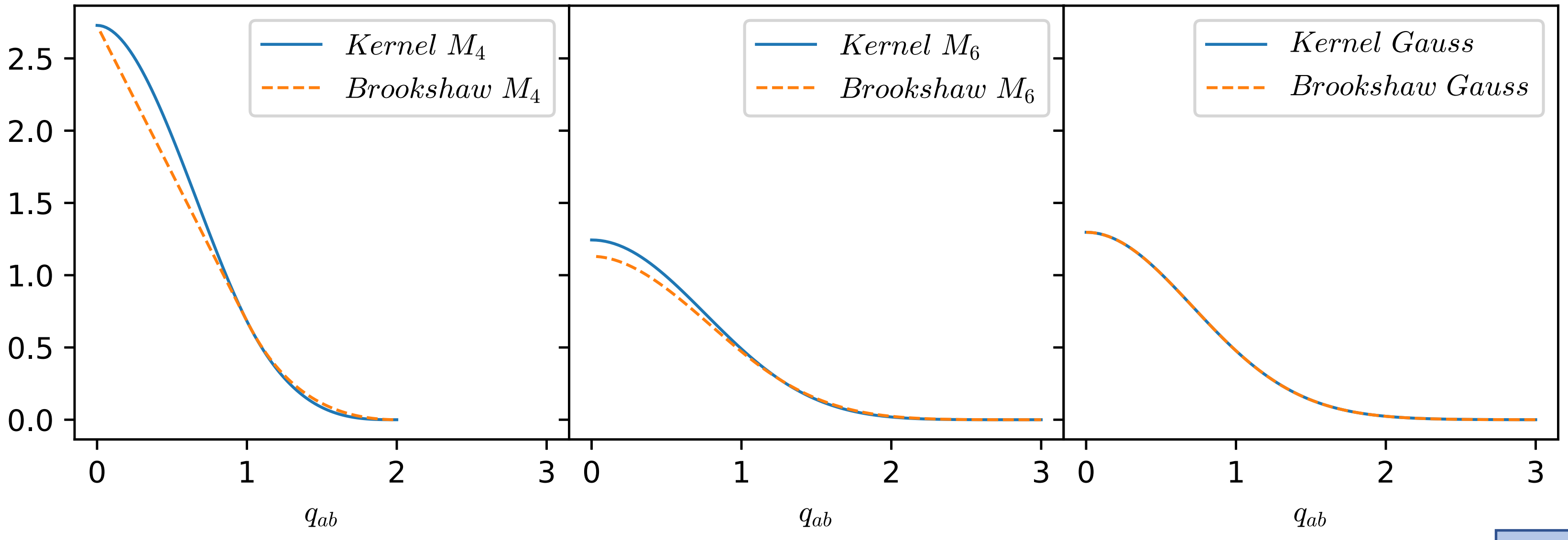
Shapes are different…

…but errors are the same.



Error terms for Laplacian operator.

Understanding Brookshaw Method



$$\nabla_a^2 W = -2\frac{(\mathbf{r}\cdot\nabla W)}{(\mathbf{r}\cdot\mathbf{r})} = \frac{-2}{C_\nu h^{\nu+2}}\frac{f'(q)}{q}$$

Brookshaw method

+

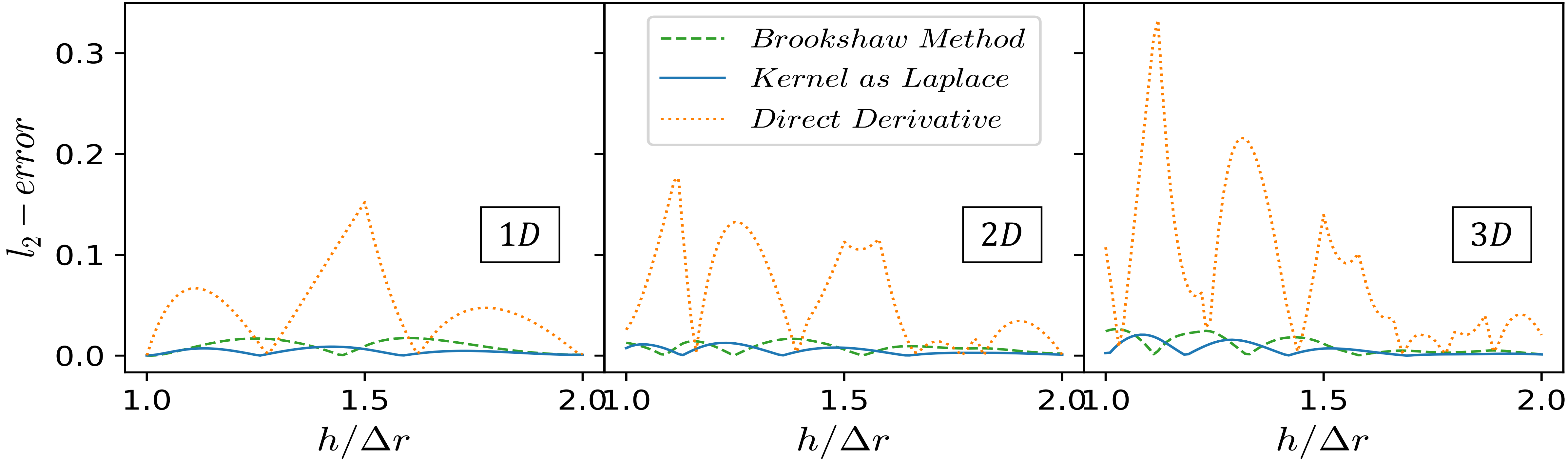
$$f(q) = \exp\left(-q^2\right)$$

With Gaussian

$$= \frac{-2}{C_\nu\ h^{\nu+2}}\frac{-2\ q\ \ \exp\left(-q^2\right)}{q} = \frac{4}{h^2}\frac{\exp\left(-q^2\right)}{C_\nu\ h^\nu} = \frac{4W_f}{h^2}$$

The kernel itself is a 2nd derivative

Isotropic Diffusion Operators



Comparison of $l_2 - error$ with regards to number of neighbors.
Cubic spline as kernel function. All dimensions.

Operators

Direct 2nd derivative

$$\frac{\partial T}{\partial t} = \sum_{ij} \frac{\partial}{\partial r^i}\left(k^{ij}\frac{\partial}{\partial r^j}T\right)$$

$$= \left(\sum_b \frac{m_b}{\rho_b} k^{ij}_{ba} \frac{\partial}{\partial r^i_a} W_{ab}\right)\left(\sum_b \frac{m_b}{\rho_b} T_{ba} \frac{\partial}{\partial r^j_a} W_{ab}\right)$$

$$+ k^{ij}_a \sum_b \frac{m_b}{\rho_b} T_{ba} \frac{\partial^2}{\partial r^i_a \partial r^j_a} W_{ab},$$

Two 1st derivatives

$$\frac{\partial T}{\partial t} = \nabla \cdot (\mathbf{k}\nabla T) \Rightarrow \frac{\partial T}{\partial t} = \nabla \cdot F$$

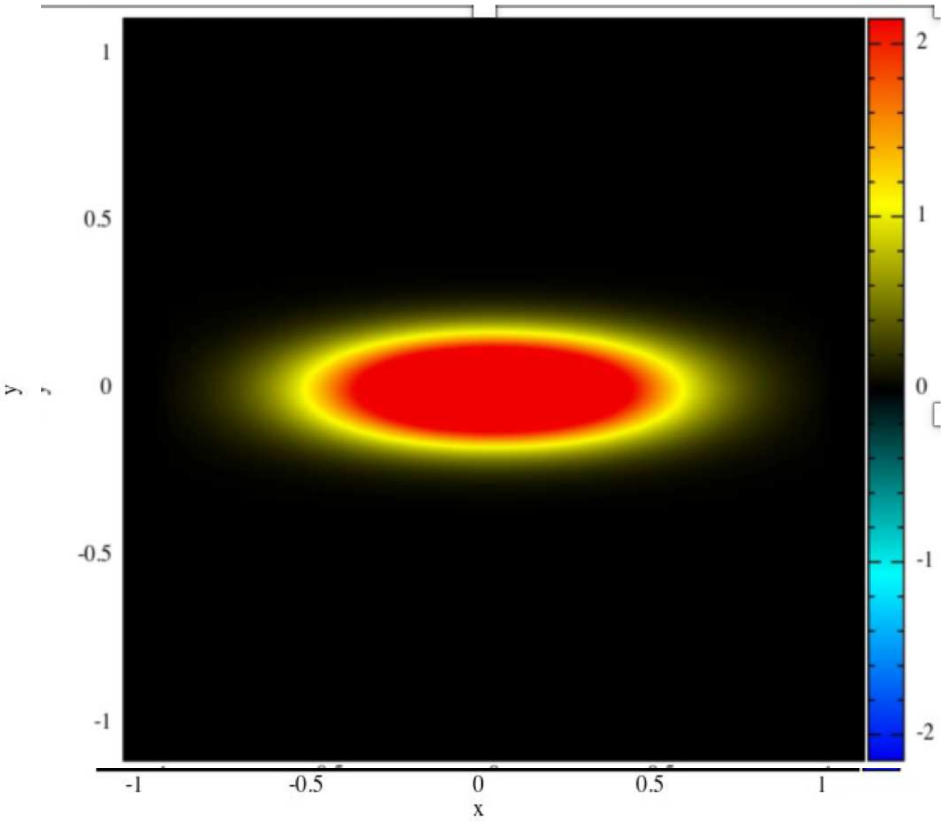
$$F = \mathbf{k} \cdot \nabla T$$

$$F = \sum_b \frac{m_b}{\rho_b} T_{ba} \nabla_a W_{ab}$$

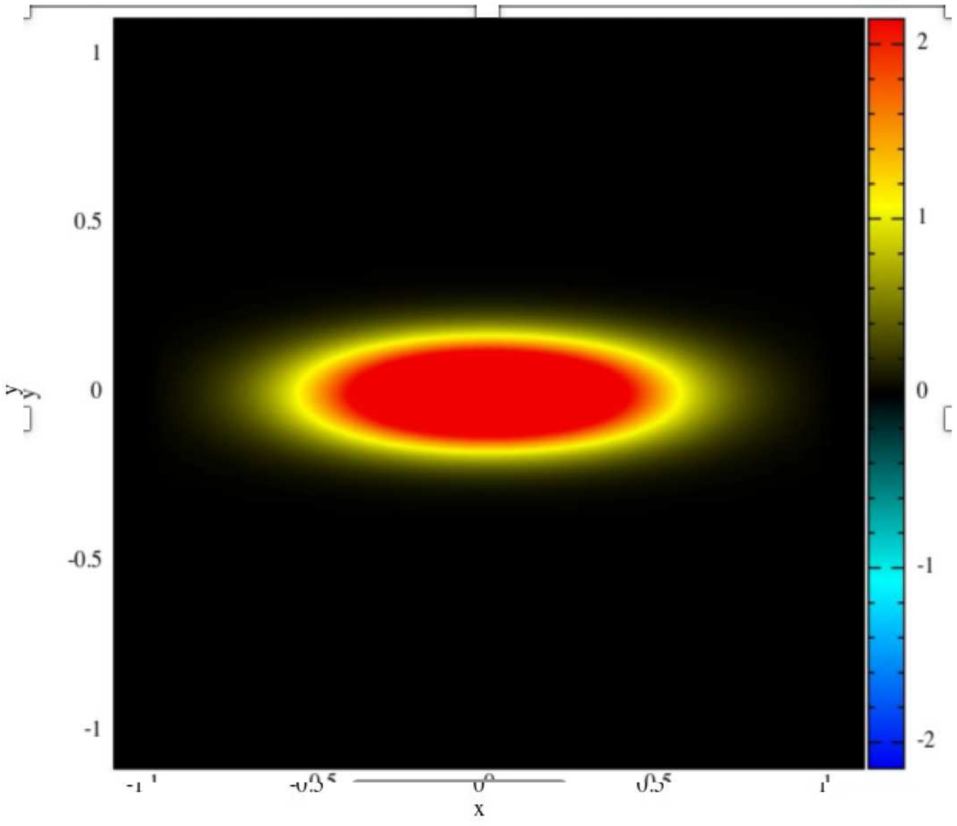
$$\frac{\partial T}{\partial t} = \sum_b m_b \left[\frac{(\mathbf{k}_a \cdot F_a)\cdot \nabla_a W_{ab}}{\Omega_a \rho_a^2} + \frac{(\mathbf{k}_b \cdot F_b)\cdot \nabla_b W_{ab}}{\Omega_b \rho_b^2}\right]$$

Diffusion with constant K

Direct 2nd derivative



Two 1st derivatives

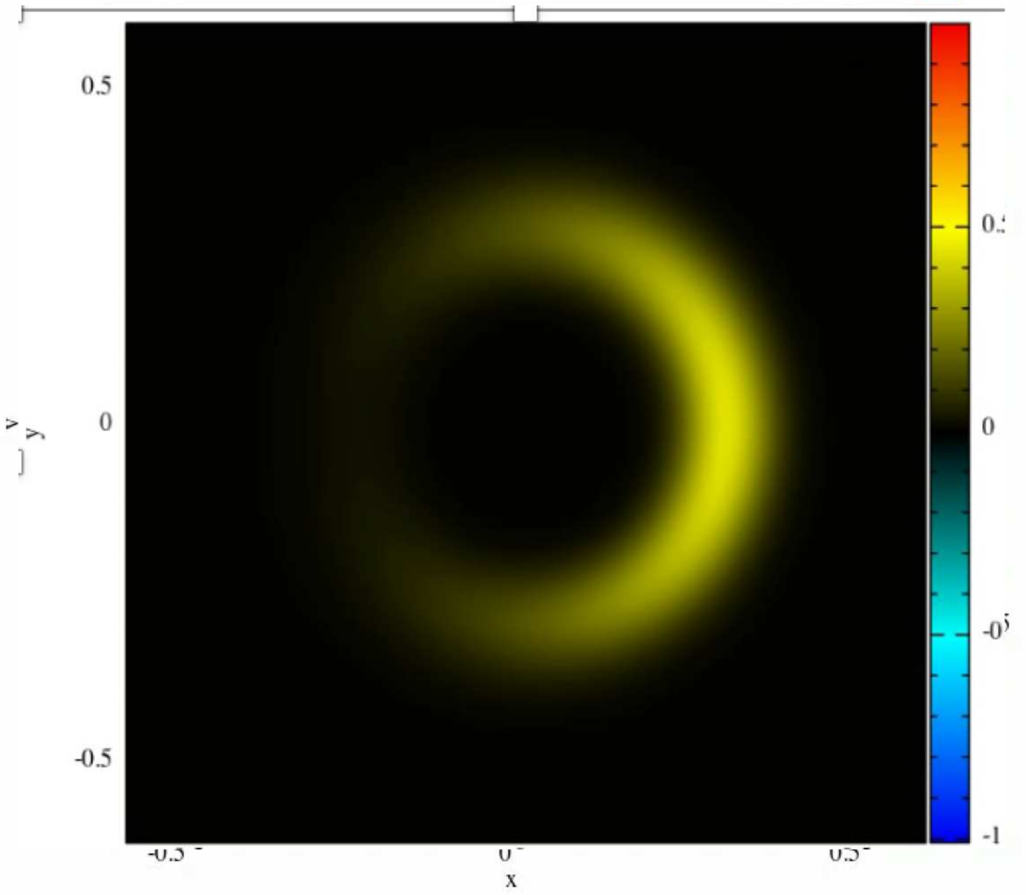


Diffusion with variable K

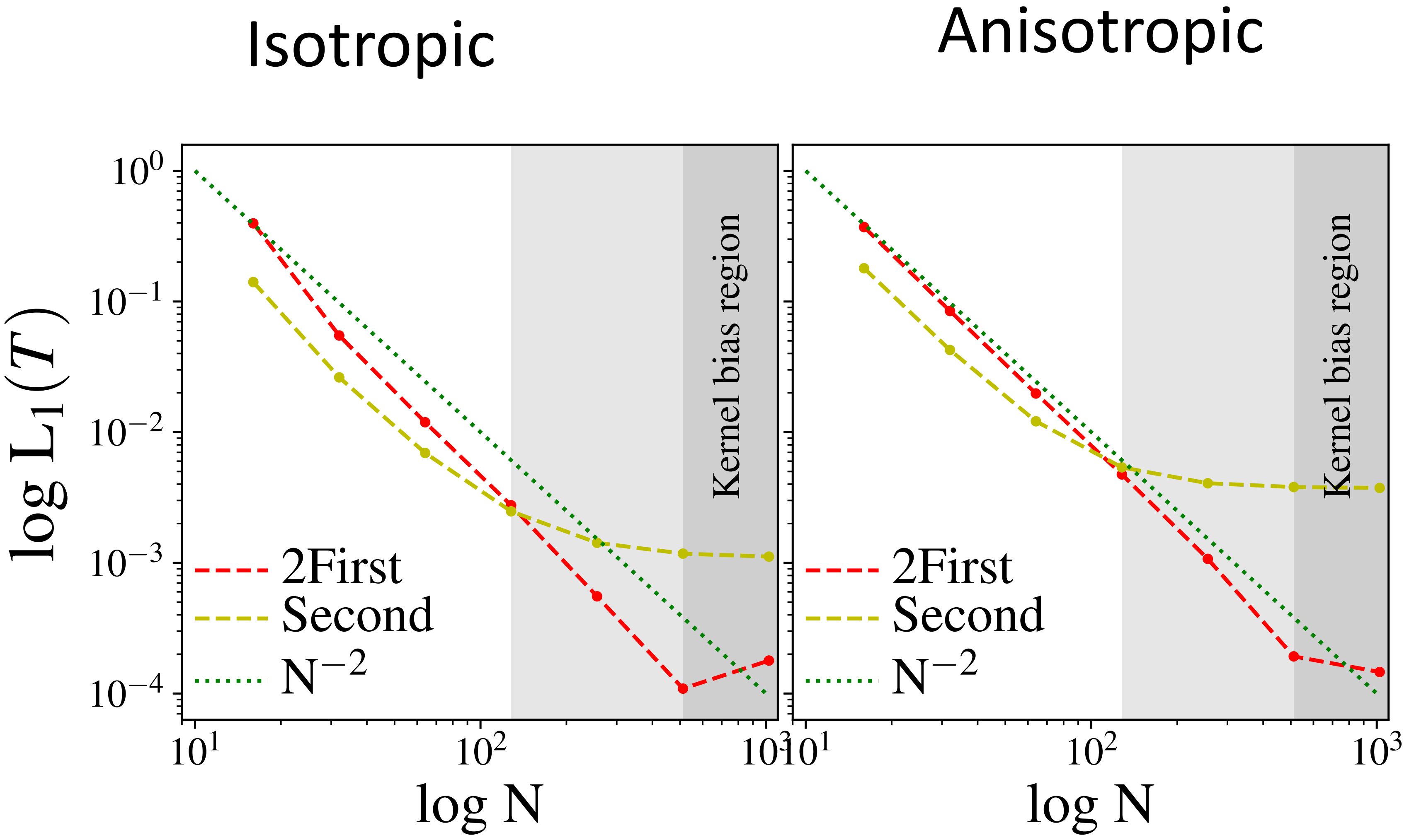
Cylindrical coordinates

$$\mathbf{k}_{\rho\phi z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \mathbf{k}_{xyz} = \frac{1}{x^2 + y^2} \begin{bmatrix} x^2 & -xy & 0 \\ -yx & y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Convergence



Summary

1. The shape of the outer part of the kernel is more important for second derivatives.

2. The idea of Brookshaw method is to mimic the kernel itself.

3. Both direct second derivative method and two first derivatives are stable for anisotropic diffusion.

4. Two first derivatives is the best method for anisotropic diffusion.