

Smoothed Particle Magnetohydrodynamics

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1 Introduction

So, you're interested in including magnetic fields in your SPH calculations? Well, let's see what is entailed.

To include magnetic fields in SPH has many fine details, but is not overly complicated. The magnetic field vector is stored directly on each particle, just as any other quantity.

2 Ideal MHD

First, let's discuss the ideal MHD equations. It is useful to understand the equations you want to solve. The equations can be summarised as follows,

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \mathbf{v}, \quad (1)$$

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} - \frac{1}{2\mu_0\rho}\nabla B^2 + \frac{1}{\mu_0\rho}(\mathbf{B} \cdot \nabla)\mathbf{B}, \quad (2)$$

$$\frac{d\mathbf{B}}{dt} = -\mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (4)$$

These are obtained as follows. Take the equations of hydrodynamics, which you know well,

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \mathbf{v}, \quad (5)$$

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho}. \quad (6)$$

These are written using the material, or Lagrangian, derivative, $d/dt = \partial/\partial t + (\mathbf{v} \cdot \nabla)$, as is appropriate for SPH.

These are combined with Maxwell's equations for electromagnetism,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (7)$$

$$\nabla \cdot \mathbf{E} = \frac{\tau}{\epsilon_0}, \quad (8)$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad (9)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (10)$$

and the Lorentz force law,

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \tau \mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (11)$$

We make a simplifying assumption here. Assume that there is an equal mix of positive and negative charges, such that macroscopically the fluid is electrically neutral. This has two consequences. One is that if the fluid is neutral, the electric field will be considerably weaker than the magnetic field. The second is that if there are an abundance of free electrons, the fluid behaves like a perfect conductor. This also implies the electric field is negligible since the electric field in the interior of a perfect conductor is zero.

This simplifies the equations we need to deal with to only consider \mathbf{B} . For example, the electric field inside a perfect conductor would not be expected to vary in time, thus we can write the current density in terms of the curl of the magnetic field,

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B}. \quad (12)$$

3 Induction equation – Evolving \mathbf{B}

Ohm's law states the current density, \mathbf{J} , as

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (13)$$

in a fixed frame of reference, where σ is the electrical conductivity of the material.

We can equate this to our earlier definition of \mathbf{J} , yielding

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma \mu_0} \nabla \times \mathbf{B} \quad (14)$$

Taking the curl of this yields

$$\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}), \quad (15)$$

where we have defined $\eta = 1/(\sigma \mu_0)$ as the magnetic resistivity. By Maxwell's equations, we can substitute out the curl of the electric field, obtaining

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B}). \quad (16)$$

We are going to deal with plasmas that are well ionised. In such a case the resistivity is effectively zero, or to put another way, the conductivity approaches infinity ($\sigma \rightarrow \text{inf}$). The induction equation is thus further simplified to

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{v} \times \mathbf{B}). \quad (17)$$

This zero resistivity is where *ideal* MHD gets its name.

If you expand out the RHS, you can write this as

$$\frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{v} \cdot \nabla) \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla) \mathbf{v} + \mathbf{v}(\nabla \cdot \mathbf{B}). \quad (18)$$

Since SPH solves equations in the co-moving reference frame, we can use these to transform the partial derivative to a Lagrangian derivative, $d/dt \equiv \partial/\partial t + (\mathbf{v} \cdot \nabla)$. Also note the

term which includes $\nabla \cdot \mathbf{B}$. This is zero in the continuum limit. In SPMHD, since it may not necessarily be zero, we “subtract” it out (that is, we do not include it as a term to be computed). The induction equation can therefore be written as

$$\frac{d\mathbf{B}}{dt} = -\mathbf{B}(\nabla \cdot \mathbf{v}) + (\mathbf{B} \cdot \nabla)\mathbf{v}. \quad (19)$$

This can also be written in terms of \mathbf{B}/ρ . Recognising that the divergence of \mathbf{v} is related to the continuity equation (Equation 5), we can write

$$\frac{d}{dt} \left(\frac{\mathbf{B}}{\rho} \right) = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla \right) \mathbf{v}. \quad (20)$$

3.1 Discretised induction equation

We can straightforwardly create an SPMHD discretised version of the induction equation. From Equation 19, we can use difference derivatives to write this as

$$\frac{d\mathbf{B}_a}{dt} = \frac{\mathbf{B}_a}{\Omega_a \rho_a} \sum_b m_b (\mathbf{v}_a - \mathbf{v}_b) \cdot \nabla_a W_{ab}(h_a) - \frac{1}{\Omega_a \rho_a} \sum_b m_b (\mathbf{v}_a - \mathbf{v}_b) \mathbf{B}_a \cdot \nabla_a W_{ab}(h_a). \quad (21)$$

In terms of \mathbf{B}/ρ , this can be written as

$$\frac{d\mathbf{B}_a}{dt} = -\frac{1}{\Omega_a \rho_a^2} \sum_b m_b (\mathbf{v}_a - \mathbf{v}_b) \mathbf{B}_a \cdot \nabla_a W_{ab}(h_a). \quad (22)$$

Both are equal in terms of efficiency. Keep in mind that almost all SPH simulations are already calculating $\nabla \cdot \mathbf{v}$, so there is no real extra cost to compute the first term in $d\mathbf{B}/dt$. You can simply multiply \mathbf{B} by the stored quantity.

In terms of accuracy, in 99.9% of situations, they will be indistinguishable. We have found *one* case where there is a difference which we traced to the manner in which the induction equation was solved. This is due to the \mathbf{B}/ρ using the density sum to compute changes due to compression/rarefaction while the other computes the velocity divergence, i.e., stemming from an integral vs derivative formulation.

4 Momentum equation – Lorentz force

The Lorentz force is

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \tau \mathbf{E} + \mathbf{J} \times \mathbf{B}. \quad (23)$$

The Lorentz force may be added to the hydrodynamic momentum equation fairly straightforwardly. Assuming the force contribution from the electric field is negligible, and remembering our definition of $\mathbf{J} = 1/\mu_0 \nabla \times \mathbf{B}$, we can write the momentum equation as

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (24)$$

The magnetic field term may be expanded using standard identities to obtain

$$\frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{2\mu_0 \rho} \nabla B^2 + \frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla) \mathbf{B}. \quad (25)$$

The first term acts as an isotropic magnetic pressure, analogous to the hydrodynamic pressure. The second term is an attractive force in the direction of the magnetic field. It behaves like a tension along magnetic field lines.

Sometimes it is useful to write the momentum equation in terms of a stress tensor,

$$\frac{dv^i}{dt} = \frac{1}{\rho} \frac{\partial S^{ij}}{\partial x^j}. \quad (26)$$

where

$$S^{ij} = -\delta^{ij} \left(P + \frac{B^2}{2\mu_0} \right) + \frac{B^i B^j}{\mu_0}. \quad (27)$$

This is of interest because the magnetic tension contains an additional force. When expanded, there is a third term,

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho} - \frac{1}{2\mu_0\rho} \nabla B^2 + \frac{1}{\mu_0\rho} (\mathbf{B} \cdot \nabla) \mathbf{B} + \frac{1}{\mu_0\rho} (\nabla \cdot \mathbf{B}) \mathbf{B}. \quad (28)$$

This term is proportional to $\nabla \cdot \mathbf{B}$, which is zero according to Maxwell's equations. Numerically, it may not. So solving the ideal MHD equations using the stress tensor can lead to difficulties.

4.1 Equations of motion from a variational pov

Just as the SPH equations of motion can be derived from the Lagrangian, so can the SPMHD equations of motion. The Lagrangian includes magnetic energy, and may be given by

$$L = \int \left(\frac{1}{2} \rho v^2 - \rho u - \frac{B^2}{2\mu_0} \right) dV. \quad (29)$$

We can replace ρdV by the mass element to obtain the discretised Lagrangian,

$$L_{\text{SPH}} = \sum_b m_b \left(\frac{1}{2} v^2 - u - \frac{B^2}{2\mu_0\rho} \right). \quad (30)$$

The action integral is stationary. That is, small perturbations must not change the solution, leading to

$$\delta S = \int \delta L_{\text{SPH}} dt = 0. \quad (31)$$

If small deviations are introduced about $\delta \mathbf{r}_a$, then we obtain

$$\delta L_a = m_a \mathbf{v}_a \cdot \delta \mathbf{v}_a - \sum_b m_b \frac{\partial u_b}{\partial \rho_b} \Big|_s \delta \rho_b + \sum_b m_b \frac{B_b^2}{2\mu_0\rho_b^2} \delta \rho_b - \sum_b m_b \frac{\mathbf{B}_b}{2\mu_0\rho_b} \cdot \delta \mathbf{B}_b. \quad (32)$$

The perturbations of $\delta \rho$ and $\delta \mathbf{B}$ can be written in terms of \mathbf{r} according to

$$\delta \rho_b = \frac{1}{\Omega_b} \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \cdot \nabla_b W_{bc}(h_b), \quad (33)$$

$$\delta \mathbf{B}_b = \frac{\mathbf{B}_b}{\Omega_b \rho_b} \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \cdot \nabla_b W_{bc}(h_b) - \frac{1}{\Omega_b \rho_b} \sum_c m_c (\delta \mathbf{r}_b - \delta \mathbf{r}_c) \mathbf{B}_b \cdot \nabla_b W_{bc}(h_b), \quad (34)$$

which come from the SPH discretised continuity and induction equations. If you insert these and solve for the resulting equation, you obtain

$$\frac{d\mathbf{v}_a}{dt} = - \sum_b m_b \left[\frac{P_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{P_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b) \right] \quad (35)$$

$$- \frac{1}{2\mu_0} \sum_b m_b \left[\frac{B_a^2}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{B_b^2}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b) \right] \quad (36)$$

$$+ \frac{1}{\mu_0} \sum_b m_b \left[\frac{\mathbf{B}_a}{\Omega_a \rho_a^2} \mathbf{B}_a \cdot \nabla_a W_{ab}(h_a) + \frac{\mathbf{B}_b}{\Omega_b \rho_b^2} \mathbf{B}_b \cdot \nabla_a W_{ab}(h_b) \right] \quad (37)$$

So, in addition the hydrodynamic pressure gradient which you are familiar with, there is a similar symmetric derivative for the magnetic pressure gradient (2nd term). The last term is the tensional force.

This formulation is equivalent to writing the momentum equation in terms of the stress tensor. We already spoke about how this includes a term proportional to $\nabla \cdot \mathbf{B}$, which numerically may not be zero. This is bundled into the third term of the preceding equation. We'll look at the consequences of this next.

4.2 Tensional force

The conservative SPMHD equations include a spurious force proportional to $\mathbf{B} \nabla \cdot \mathbf{B}$. This can be problematic when the divergence of the magnetic field is large or when the magnetic field itself is strong. Stability analysis shows that this force causes SPMHD to become tensile unstable when plasma $\beta \equiv P/(B^2/2\mu_0) < 1$, that is, when the magnetic pressure is dominant over the hydrodynamic pressure.

Here is an example from a simulation of the MHD rotor test case. It should spin, but this spurious force causes the particle to attract each other, overpowering the hydrodynamic and magnetic pressure.

There are two approaches to solving this problem, outlined below.

4.2.1 Solutions to the tensile instability

One is to approach this in terms of the low order estimate of our tensional force inherent with the symmetric derivative. Using a difference derivative instead will be more accurate, and as it turns out, remains stable for most everything. This is the approach utilised by Morris (1996), so it is often called the Morris method.

The magnetic pressure is calculated using the conservative symmetric derivative, but the tensional force is calculated according to

$$\frac{dv_a^i}{dt} = - \frac{1}{\mu_0 \Omega_a \rho_a} \sum_b \frac{m_b}{\rho_b} (B^i B^j)_a - (B^i B^j)_b \frac{\partial W_{ab}(h_a)}{\partial x^j}. \quad (38)$$

While this will stabilise a calculation for strong magnetic fields, it does come with the cost of no longer conserving momentum. This can be extreme in some circumstances.

A second approach is by Borge et al (2001), whereby they explicitly subtract out the spurious tensional force from the conservative force. They compute the $\mathbf{B} \nabla \cdot \mathbf{B}$ term using

a symmetric derivative,

$$\mathbf{B}_a \nabla \cdot \mathbf{B}_a = \mathbf{B}_a \sum_b m_b \left[\frac{\mathbf{B}_a}{\Omega_a \rho_a^2} \cdot \nabla_a W_{ab}(h_a) + \frac{\mathbf{B}_b}{\Omega_b \rho_b^2} \cdot \nabla_a W_{ab}(h_b) \right]. \quad (39)$$

Subtracting this term out also provides stable calculations in all circumstances. Technically, since the instability sets in when $\frac{1}{2}B^2 > P$, only $\frac{1}{2}\mathbf{B}\nabla \cdot \mathbf{B}$ need be subtracted to achieve stability. However, this can lead to circumstances where the particles are left in a near-pressureless state causing significant particle disorder.

In Phantom, we use the Borge subtraction method, but also include a switch, $f\mathbf{B}\nabla \cdot \mathbf{B}$, so that the correction is not applied when the magnetic field is subdominant. The correction is ramped up to the full subtraction as plasma $\beta \rightarrow 1$. The functional form is given by

$$f = \begin{cases} 1 & \beta < 1, \\ 2 - \beta & 1 \leq \beta < 2, \\ 0 & \beta \geq 2. \end{cases} \quad (40)$$

This provides a compromise between conservation of momentum and stability.

5 The divergence-free constraint – $\nabla \cdot \mathbf{B} = 0$

It is vitally important to maintain the divergence-free constraint of the magnetic field. The problem is that $\nabla \cdot \mathbf{B}$ is an initial condition for the magnetic field, but not manifestly enforced. In the continuum limit, a divergence-free magnetic field stays divergence-free. Not necessarily so numerically.

Minimising $\nabla \cdot \mathbf{B}$ can significantly help with momentum conservation.

We use the constrained divergence cleaning technique. The general idea is that you continually remove divergence errors from the magnetic field as they are generated. Two approaches work in concert. One is to propagate divergence errors away from their source. This usually makes sense since the locations in your simulation that are generating errors are probably the most dynamical, therefore of most interest. The second is to use a diffusion term targeted on the divergence of the magnetic field to remove errors completely from the domain.

The equations solved include a new scalar field, ψ , to facilitate the cleaning. The equations are

$$\frac{d\mathbf{B}}{dt} = -\nabla\psi, \quad (41)$$

$$\frac{d}{dt} \left(\frac{\psi}{c_h} \right) = -c_h \nabla \cdot \mathbf{B} - \frac{1}{\tau} \frac{\psi}{c_h} - \frac{1}{2} \frac{\psi}{c_h} \nabla \cdot \mathbf{v}. \quad (42)$$

c_h represents the speed at which divergence errors are transported away from their source, and is typically set to the fast MHD wave speed.

The rate of dissipation of divergence error is governed by $\tau = h/\sigma c_h$. In 3D, we use the empirically derived value of $\sigma = 1$.

To obtain the SPMHD discretised equations, we can define the energy content of ψ to be

$$e_\psi = \frac{\psi^2}{2\mu_0 \rho c_h^2}. \quad (43)$$

If you include this in the total energy,

$$E = \int \left[\frac{1}{2} \rho v^2 + \rho u + \frac{B^2}{2\mu_0} + e_\psi \right] dV, \quad (44)$$

you can use the constraint of energy conservation to obtain SPMHD equations.

The SPMHD discretised version of these equations require conjugate symmetric/difference operators for $\nabla\psi$ and $\nabla \cdot \mathbf{B}$. They are derived by defining an energy for the ψ field and including that as part of the Lagrangian, which then dictates how to construct conservative forms of these equations.

The discretised equations are

$$\frac{d\mathbf{B}_a}{dt} = -\rho_a \sum_b m_b \left[\frac{\psi_a}{\Omega_a \rho_a^2} \nabla_a W_{ab}(h_a) + \frac{\psi_b}{\Omega_b \rho_b^2} \nabla_a W_{ab}(h_b) \right], \quad (45)$$

$$\frac{d}{dt} \left(\frac{\psi_a}{c_{h,a}} \right) = \frac{c_{h,a}}{\Omega_a \rho_a} \sum_b m_b (\mathbf{B}_a - \mathbf{B}_b) \cdot \nabla W_{ab}(h_a) - \frac{1}{\tau} \frac{\psi}{c_{h,a}} + \frac{1}{2} \frac{\psi}{c_{h,a}} \sum_b m_b (\mathbf{v}_a - \mathbf{v}_b) \cdot \nabla_a W_{ab}(h_a). \quad (46)$$

Note that for $d/dt(\psi/c_h)$, the quantities of $\nabla \cdot \mathbf{B}$ and $\nabla \cdot \mathbf{v}$ are typically already calculated in the code. Thus, this method is quite efficient as it imposes no additional timestep constraints, and only requires computing an additional symmetric derivative for $\nabla\psi$.