Physical and Artificial Resistivity
(in smoothed particle magnetohydrodynamics)

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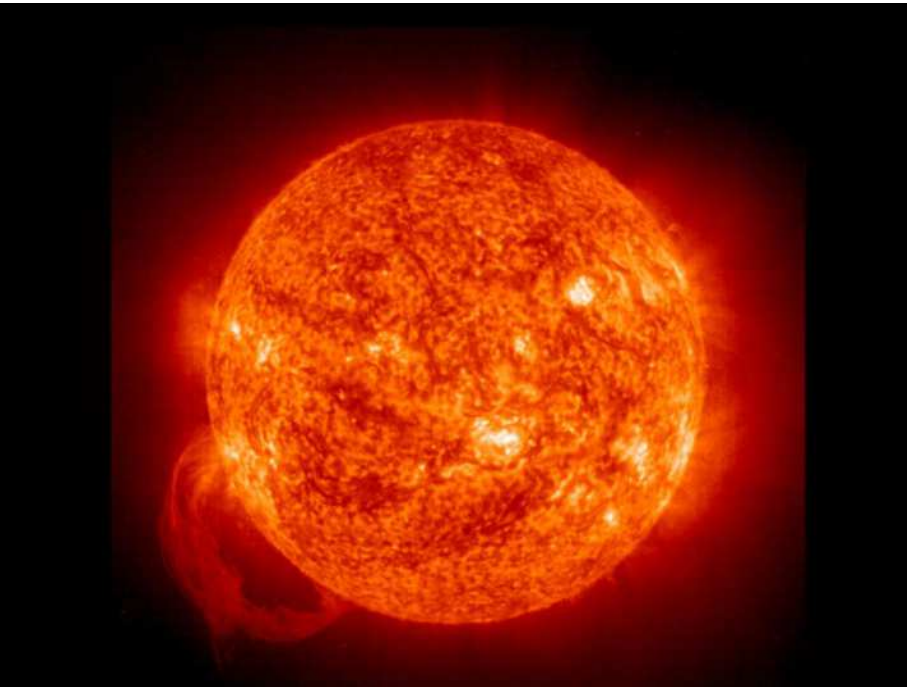
1st Phantom Users Workshop
Monash University, 20 February 2018

Ideal magnetohydrodynamics

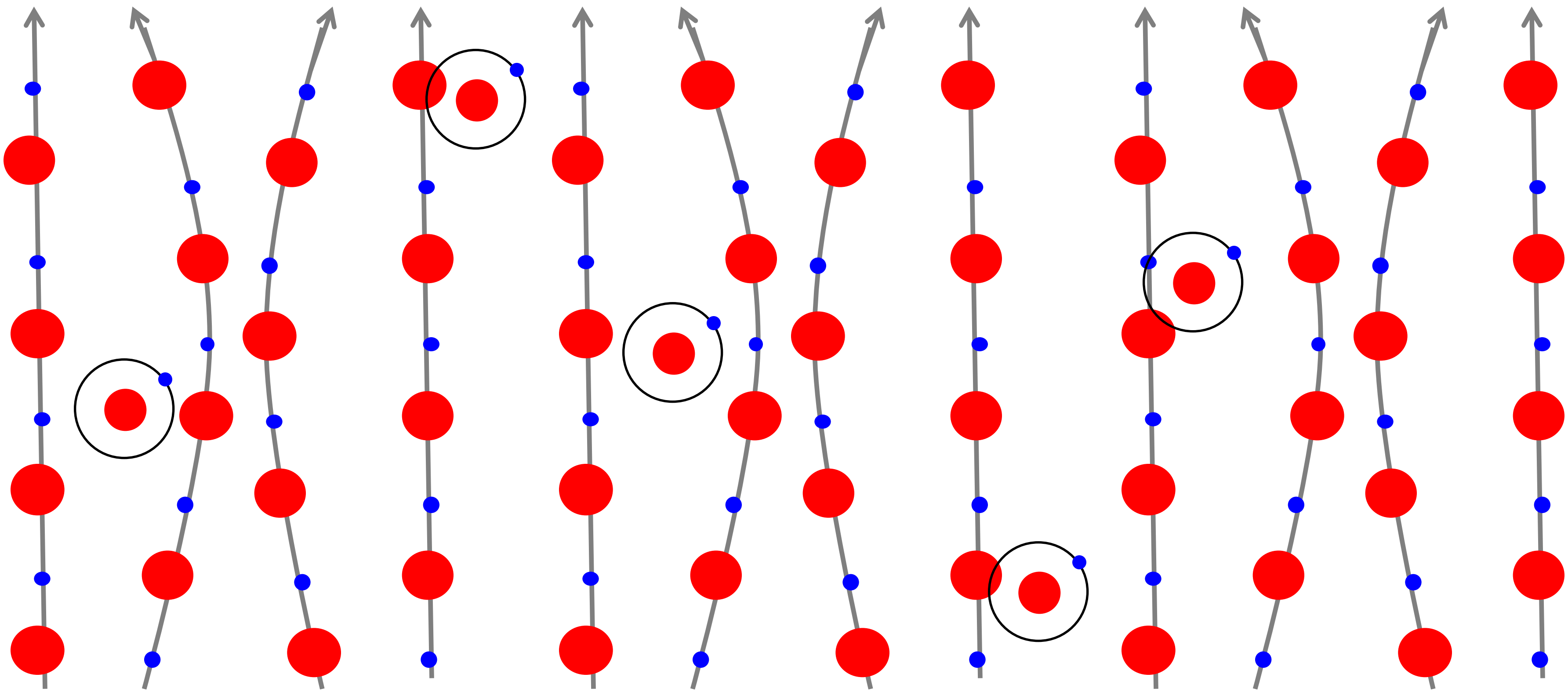


$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} - \boldsymbol{B}\,(\boldsymbol{\nabla}\cdot\boldsymbol{v})$$

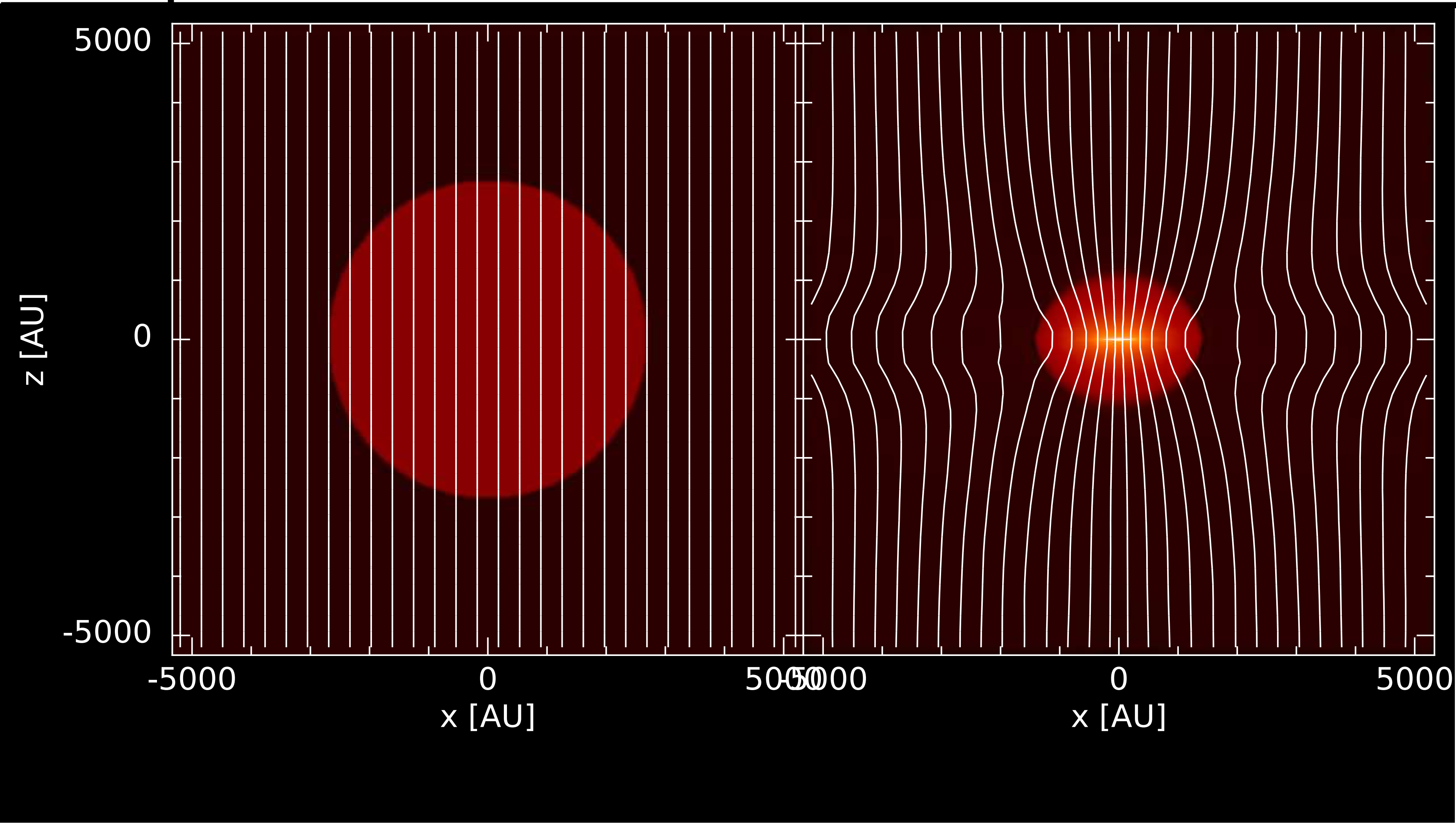
Ideal MHD

- Fully ionised plasma 

- Zero resistivity & infinite conductivity
- Ions & electrons are tied to the magnetic field

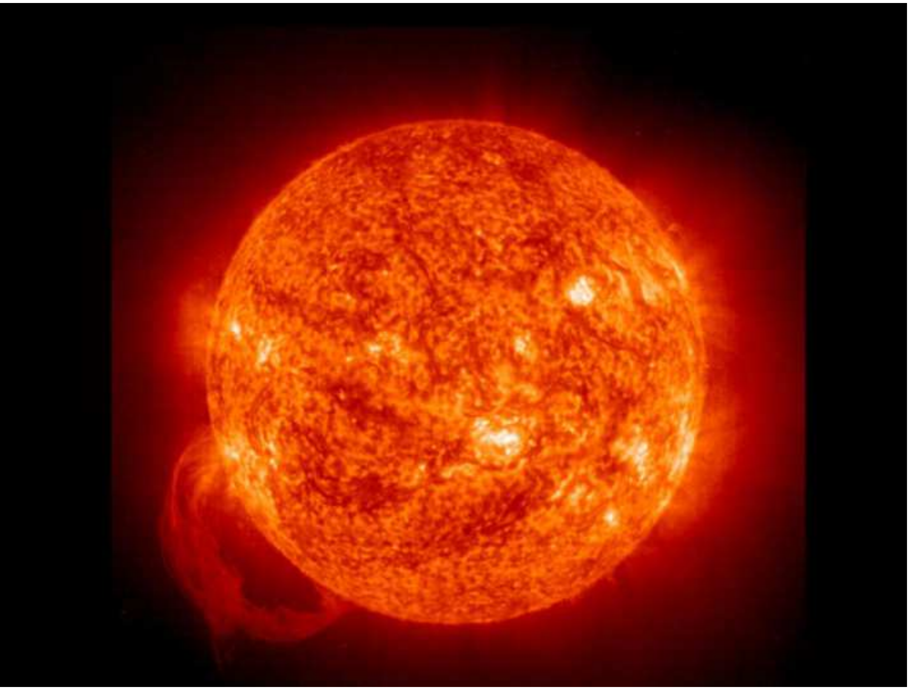


Ideal MHD



Density (rendered) + Magnetic field lines
Ideal MHD. Left: Initial conditions. Right: at $\rho_{max} = 10^{-9}$g cm^{-3}

Ideal MHD: Artificial Resistivity



$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{v} - \boldsymbol{B}\,(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + \boldsymbol{\nabla}\times\eta_{\mathrm{art}}\,(\boldsymbol{\nabla}\times\boldsymbol{B})$$

where

$$\eta_{\mathrm{art}} \approx \frac{1}{2}\alpha_{\mathrm{B}} v_{\mathrm{sig}} h$$

Ideal MHD: Artificial Resistivity

- Artificial resistivity (Tricco & Price, 2013)

$$\frac{\mathrm{d}B_a^i}{\mathrm{d}t} = -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left[v_{ab}^i B_a^j \nabla_a^j W_{ab}(h_a) - B_a^i v_{ab}^j \nabla_a^j W_{ab}(h_a) \right] + \left. \frac{\mathrm{d}B_a^i}{\mathrm{d}t} \right|_{\mathrm{art}}$$

$$\left. \frac{\mathrm{d}B_a^i}{\mathrm{d}t} \right|_{\mathrm{art}} = \frac{\rho_a}{2} \sum_b m_b B_{ab}^i \left[\frac{\alpha_a^{\mathrm{B}} v_{\mathrm{sig},a} \hat{r}_{ab}^j \nabla_a^j W_{ab}(h_a)}{\Omega_a \rho_a^2} + \frac{\alpha_b^{\mathrm{B}} v_{\mathrm{sig},b} \hat{r}_{ab}^j \nabla_a^j W_{ab}(h_b)}{\Omega_b \rho_b^2} \right]$$

$$v_{ab}^i = v_a^i - v_b^i$$

$$B_{ab}^i = B_a^i - B_b^i$$

$$v_{\mathrm{sig},a} = \sqrt{c_{\mathrm{s},a}^2 + v_{\mathrm{A},a}^2}$$

$$\alpha_a^{\mathrm{B}} = \min\left(\frac{h_a \left| \nabla \boldsymbol{B}_a \right|}{\left| \boldsymbol{B}_a \right|}, 1 \right)$$

$$\left| \nabla \boldsymbol{B}_a \right| \equiv \sqrt{\sum_i \sum_j \left| \frac{\partial B_a^i}{\partial x_a^j} \right|^2}$$

- Always applied if there is a gradient in the magnetic field (i.e. $|\nabla \mathbf{B}| > 0$)

Ideal MHD: Artificial Resistivity

- Artificial resistivity (Price, et al, submitted)

$$
\begin{aligned}
\frac{\mathrm{d}B_a^i}{\mathrm{d}t} &= -\frac{1}{\Omega_a \rho_a} \sum_b m_b \left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right) \right] + \left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}} \\
\left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\mathrm{art}} &= \frac{\rho_a}{2} \sum_b m_b \alpha^{\mathrm{B}} v_{\mathrm{sig},ab} B_{ab}^i \left[\frac{\hat{r}_{ab}^j \nabla_a^j W_{ab}(h_a)}{\Omega_a \rho_a^2} + \frac{\hat{r}_{ab}^j \nabla_a^j W_{ab}(h_b)}{\Omega_b \rho_b^2} \right] \\
B_{ab}^i &= B_a^i - B_b^i \\
v_{\mathrm{sig},ab} &= \left| \boldsymbol{v}_{ab} \times \hat{\boldsymbol{r}}_{ab} \right| \\
\alpha^{\mathrm{B}} &\equiv 1
\end{aligned}
$$

  - Always applied for non-zero velocity
  - Less resistive that that from Tricco & Price (2013)

Ideal MHD: Artificial Resistivity

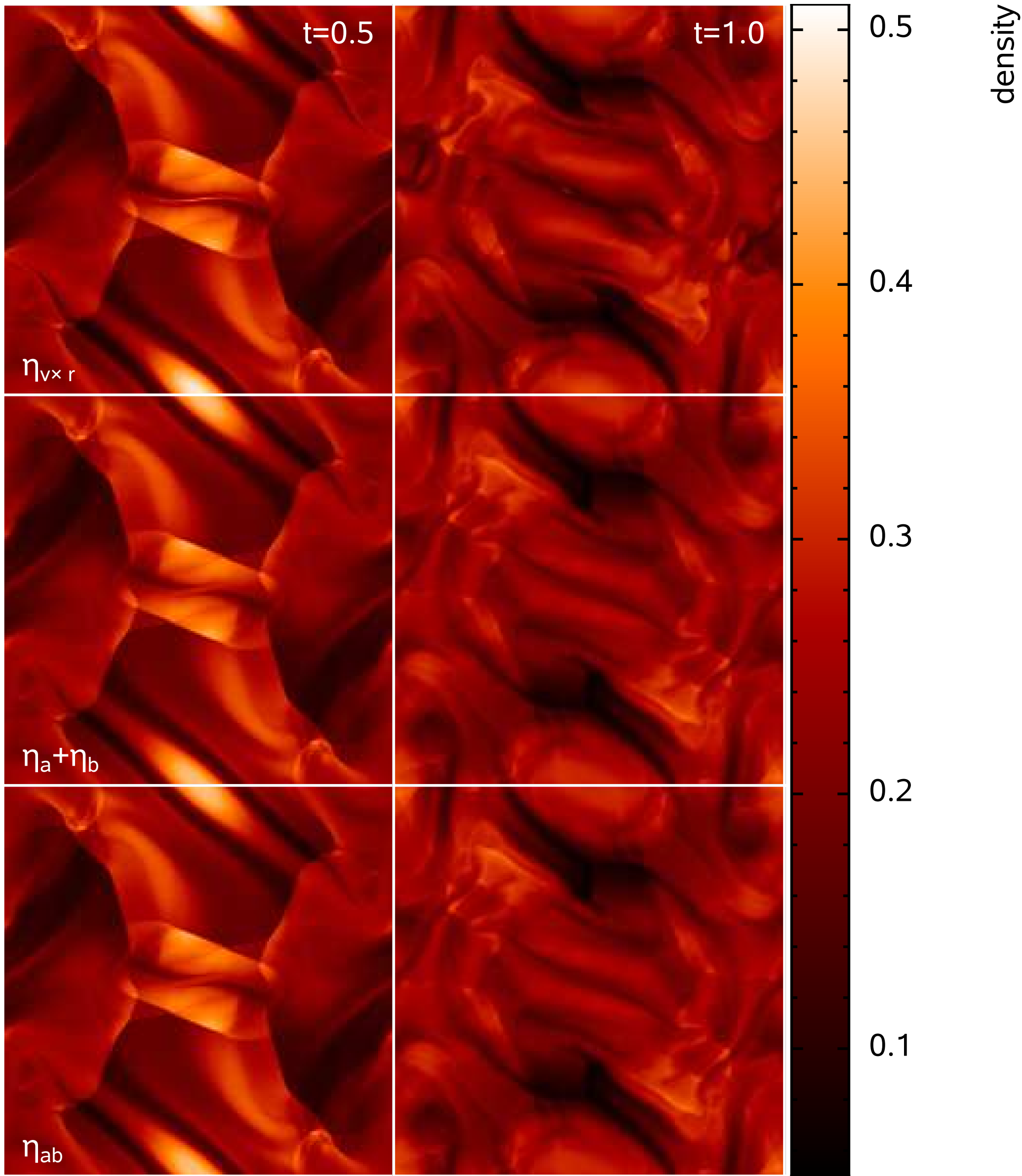
- Price et. al. (2017) artificial resistivity

$$v_{\mathrm{sig},ab} = |\boldsymbol{v}_{ab} \times \hat{\boldsymbol{r}}_{ab}|$$
$$\alpha^{\mathrm{B}} \equiv 1$$

- Tricco & Price (2013)

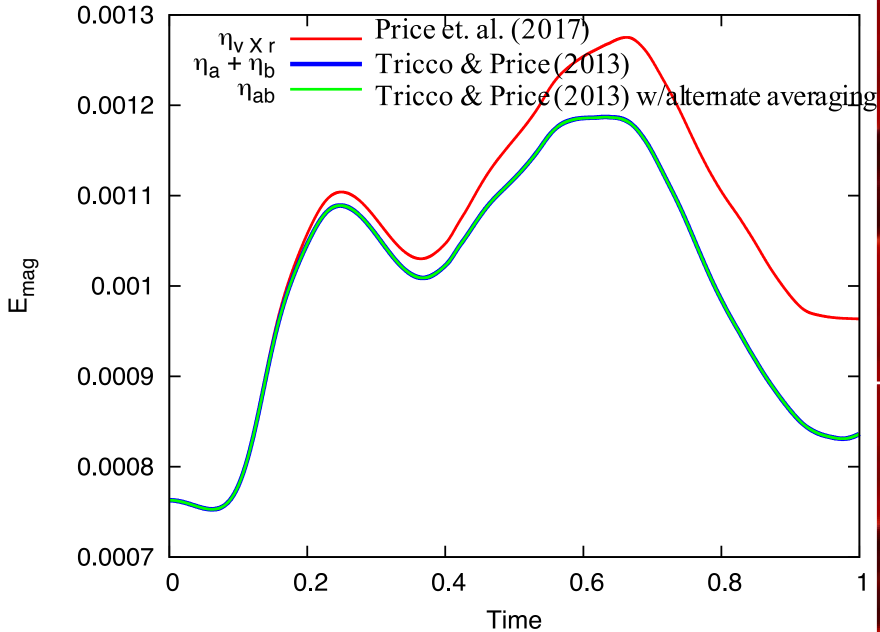
$$v_{\mathrm{sig},a} = \sqrt{c_{\mathrm{s},a}^2 + v_{\mathrm{A},a}^2}$$
$$\alpha_a^{\mathrm{B}} = \min\left(\frac{h_a\,|\nabla \boldsymbol{B}_a|}{|\boldsymbol{B}_a|}, 1\right)$$

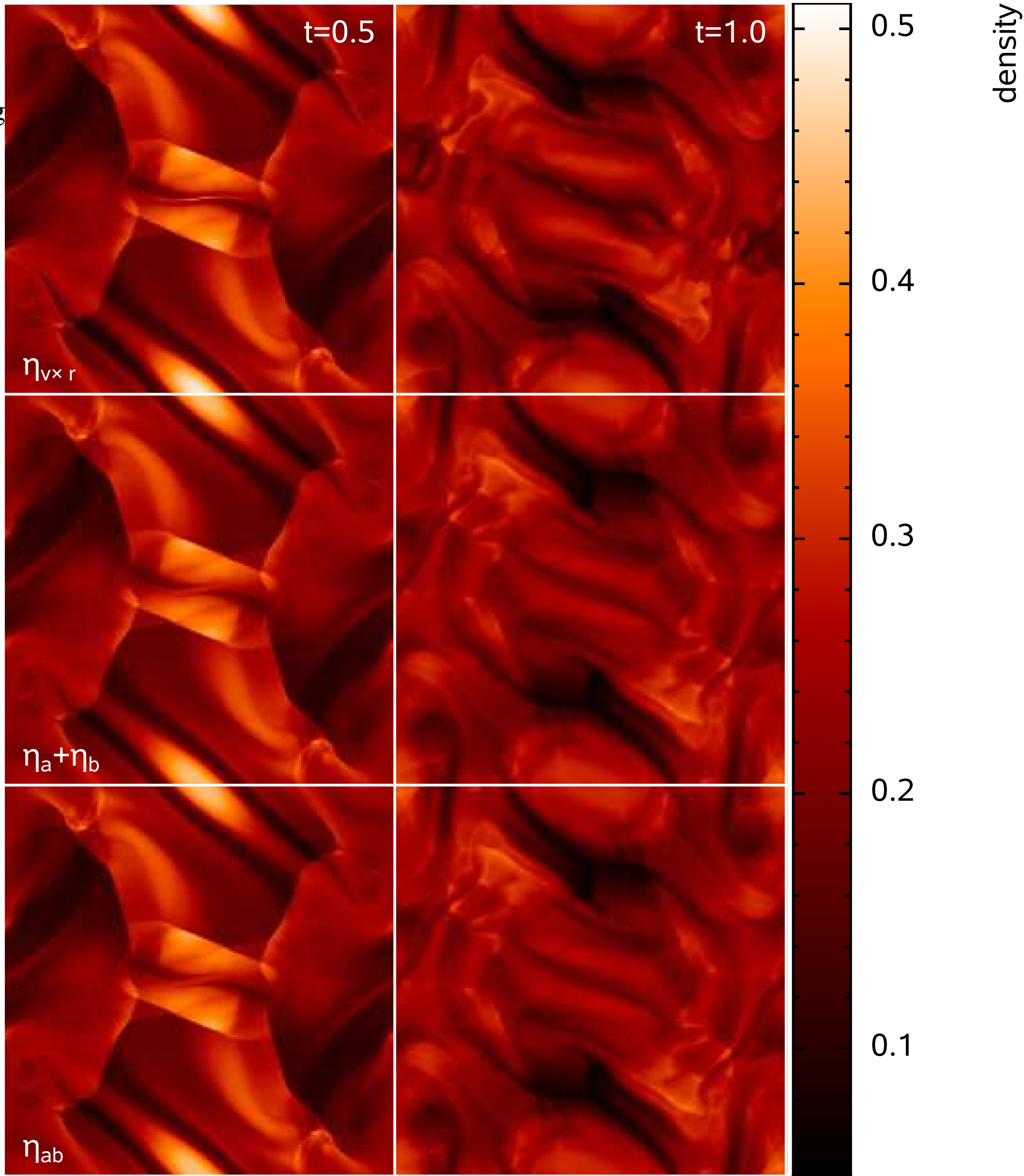
- Tricco & Price (2013) with alternate averaging



Wurster, Bate, Price & Tricco (2017)

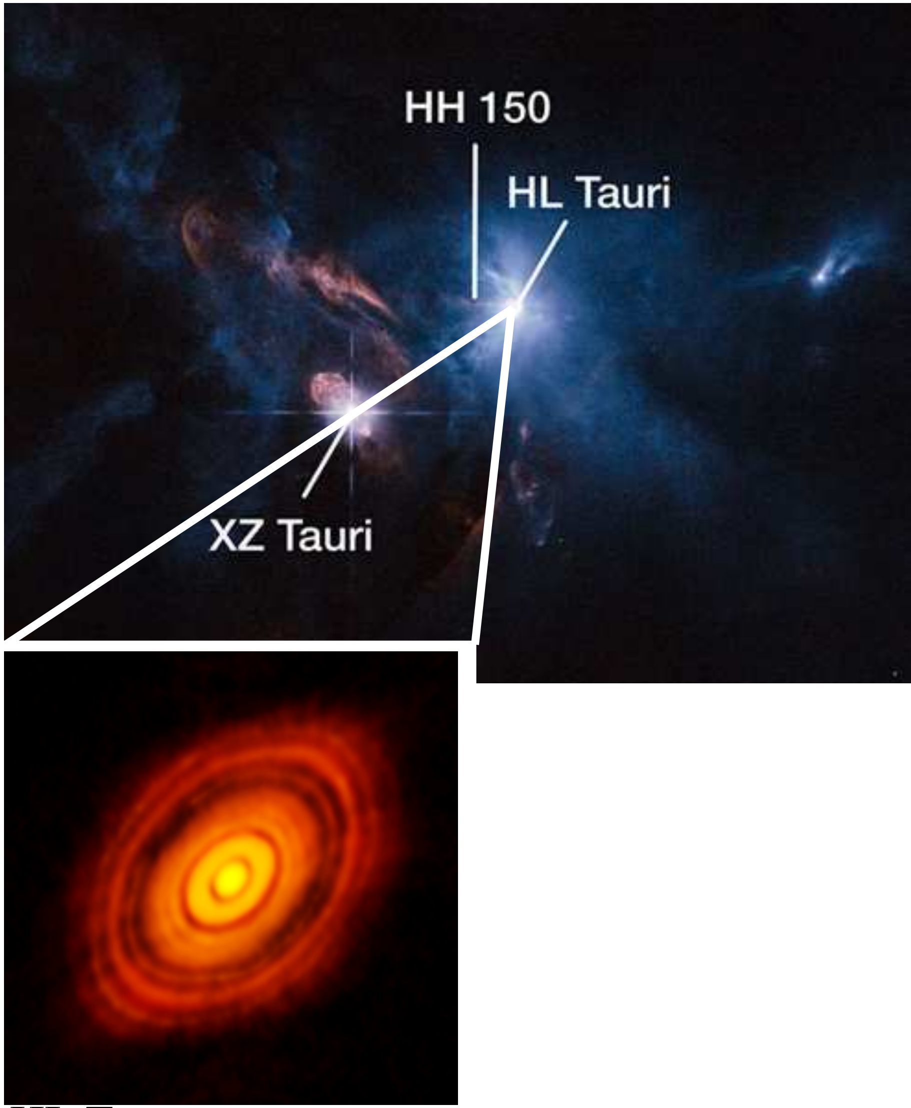
Ideal MHD: Artificial Resistivity





Wurster, Bate, Price & Tricco (2017)

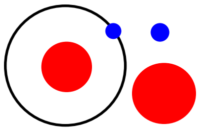
Motivation: Non-ideal MHD

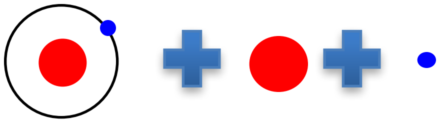


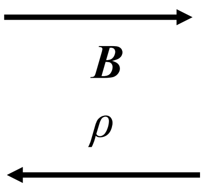
Orion Molecular Cloud
Ionisation fraction ~ 10^{-14}

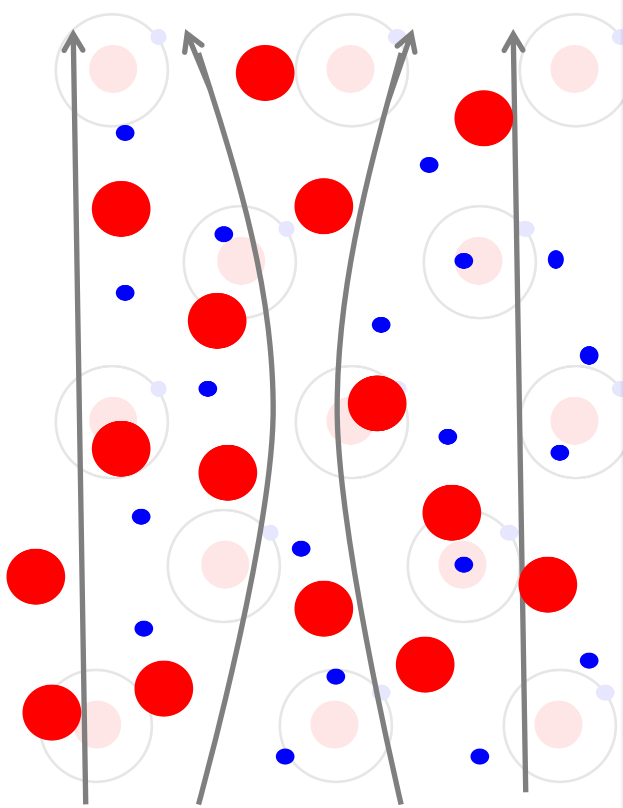
HL Tau
~ 10^{-12}

Non-ideal MHD

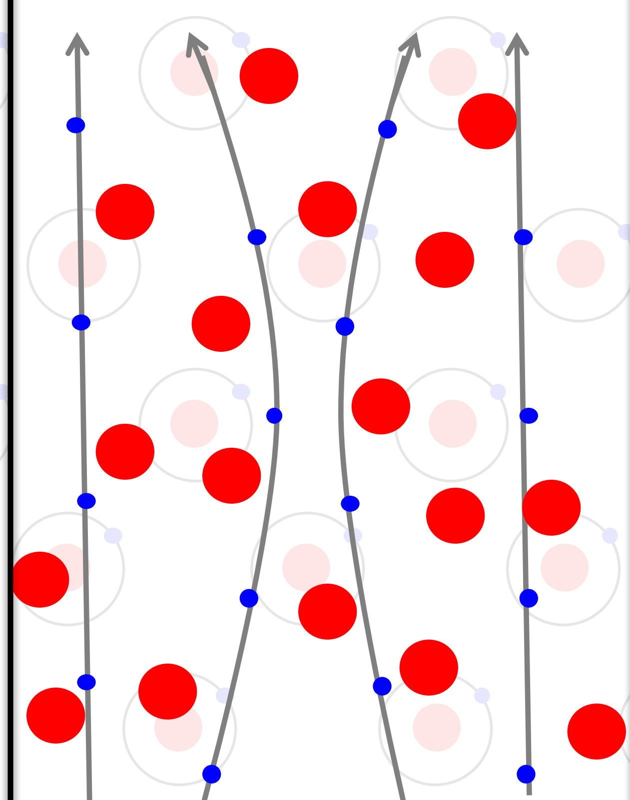


- ➢Partially ionised plasma 
- ➢Non-zero resistivity & conductivity
- ➢Ions, electrons & neutrals behaviour is environment-dependent

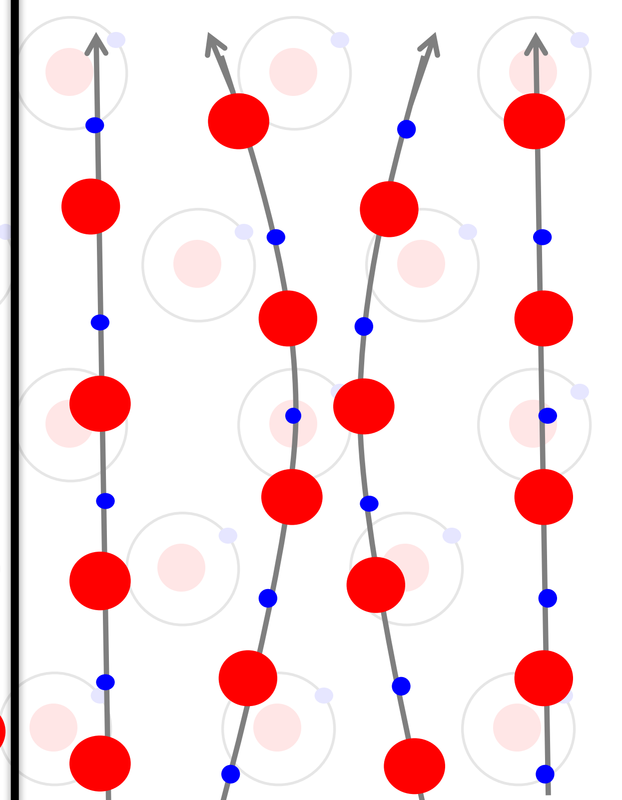




Ohmic Resistivity

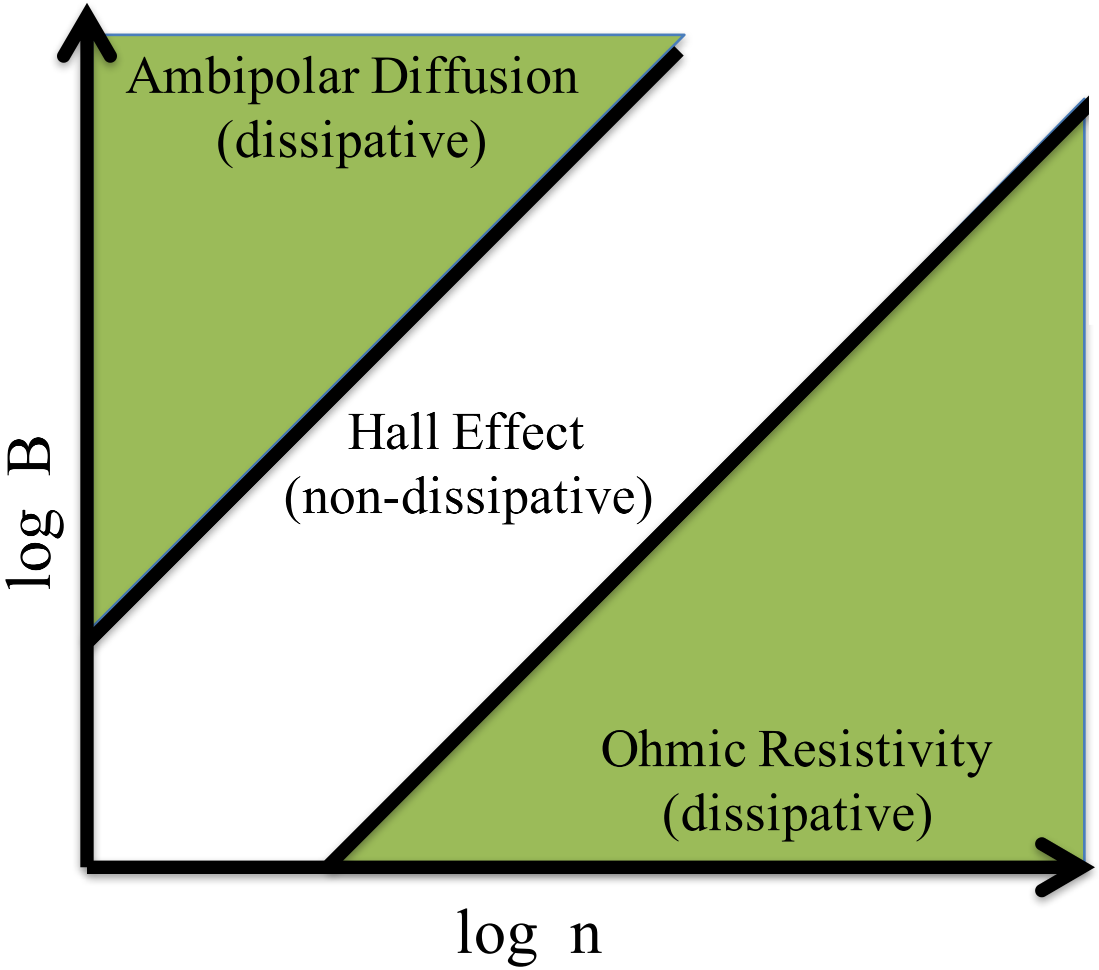


Hall Effect
(ion-electron drift)



Ambipolar Diffusion
(ion-neutral drift)

Non-ideal MHD



$$\left.\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right|_{\mathrm{OR}} = -\boldsymbol{\nabla} \times \eta_{\mathrm{OR}} \left(\boldsymbol{\nabla} \times \boldsymbol{B}\right),$$

$$\left.\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right|_{\mathrm{HE}} = -\boldsymbol{\nabla} \times \eta_{\mathrm{HE}} \left[\left(\boldsymbol{\nabla} \times \boldsymbol{B}\right) \times \hat{\boldsymbol{B}}\right],$$

$$\left.\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right|_{\mathrm{AD}} = \boldsymbol{\nabla} \times \eta_{\mathrm{AD}} \left\{\left[\left(\boldsymbol{\nabla} \times \boldsymbol{B}\right) \times \hat{\boldsymbol{B}}\right] \times \hat{\boldsymbol{B}}\right\}.$$

Adapted from Wardle (2007)

Non-ideal MHD: Hall effect

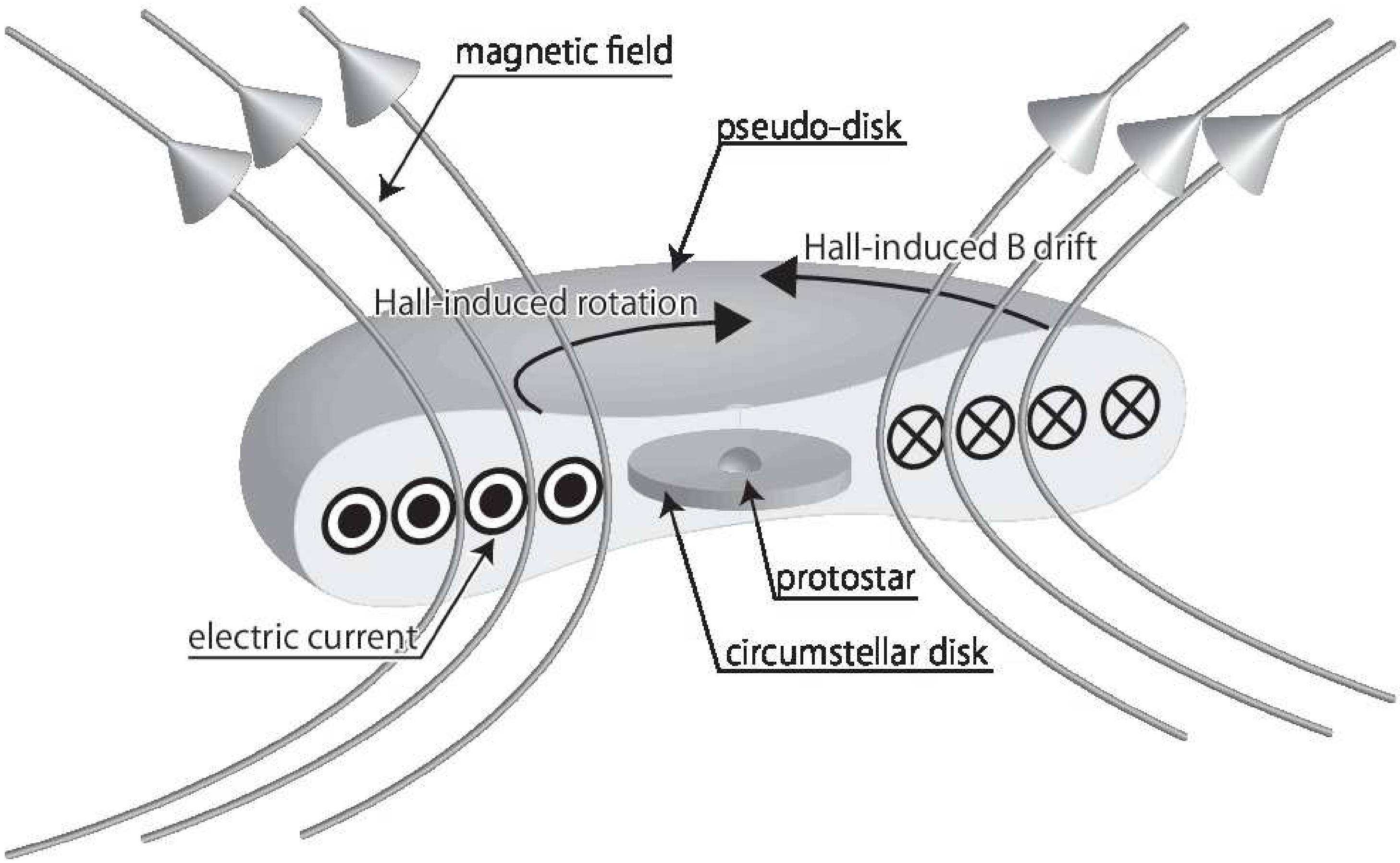
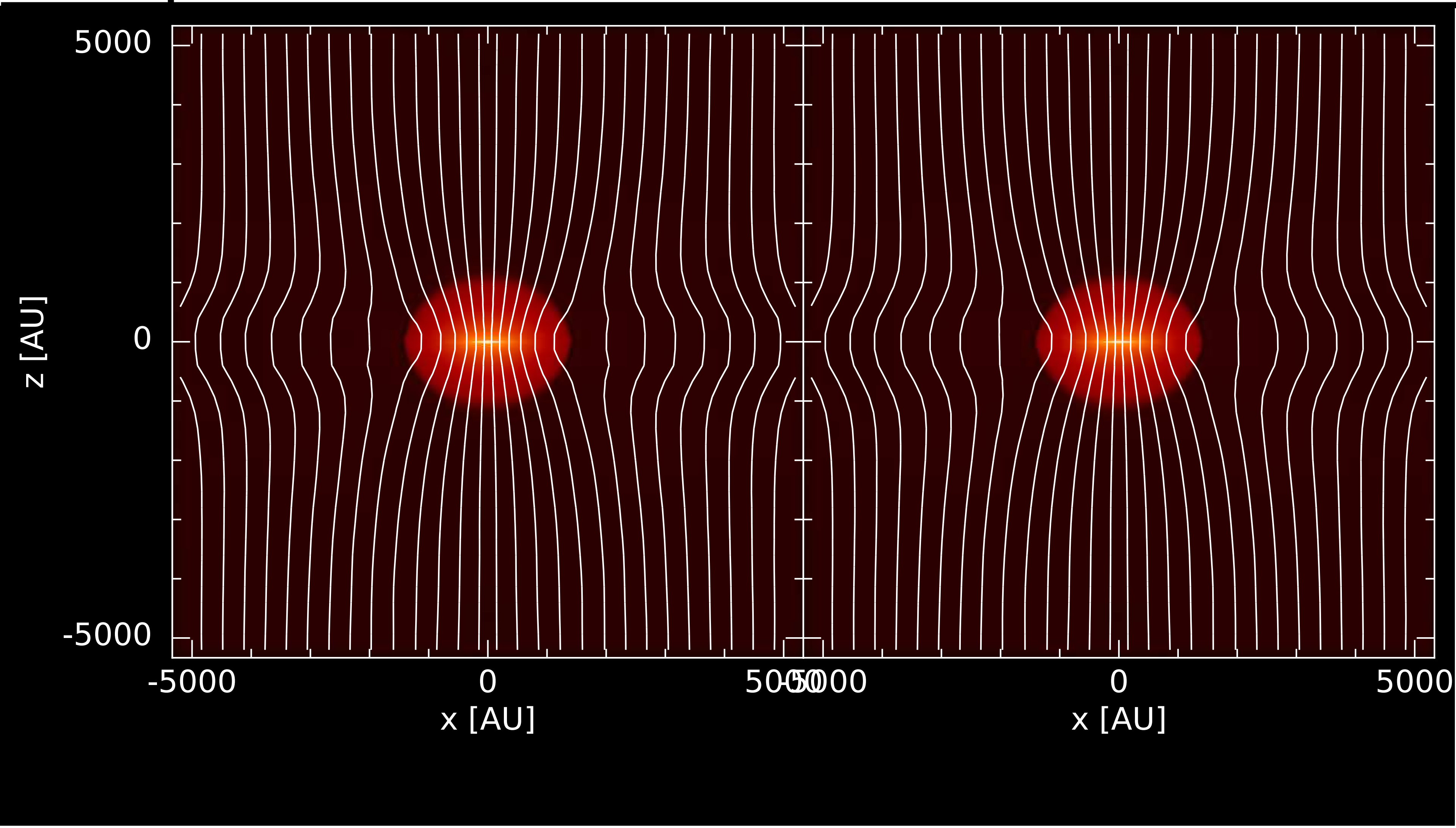


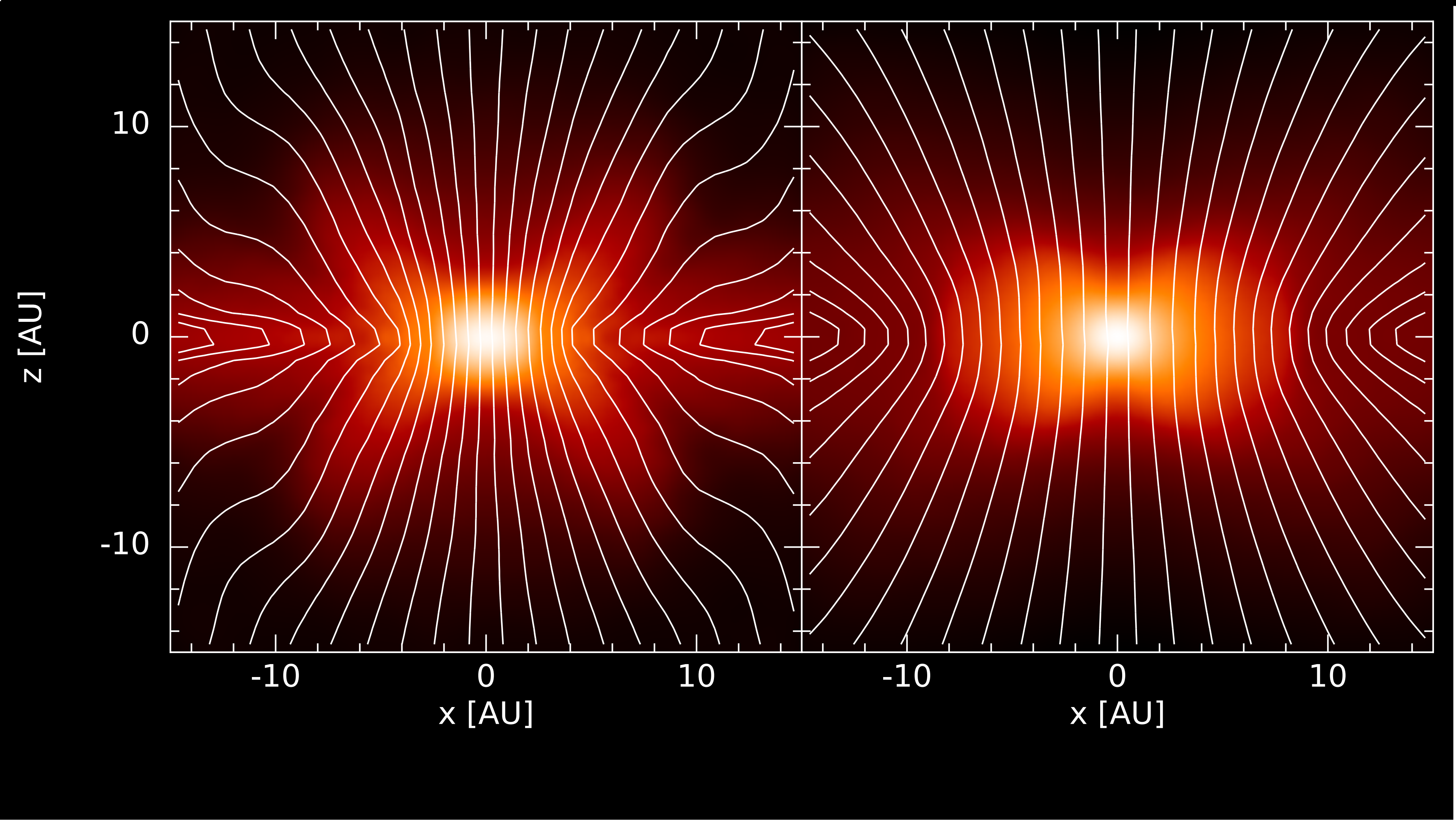
Image credit: Tsukamoto et al (2017); see also: Braiding & Wardle (2012a,b)

Ideal vs non-ideal MHD



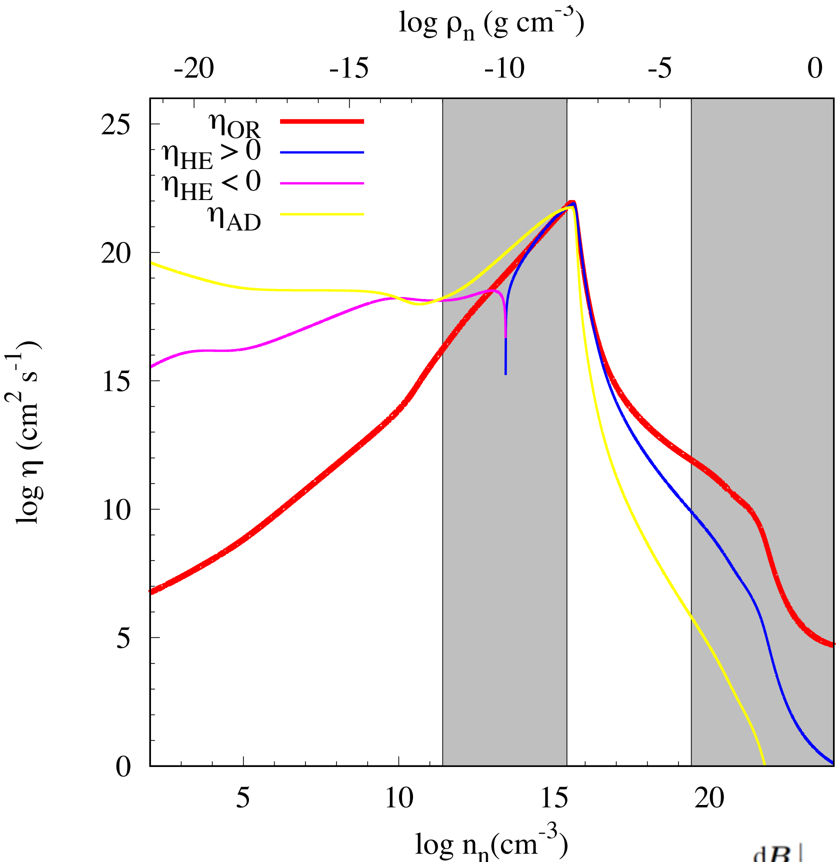
Density (rendered) + Magnetic field lines
During first core phase. Left: ideal MHD. Right: non-ideal MHD

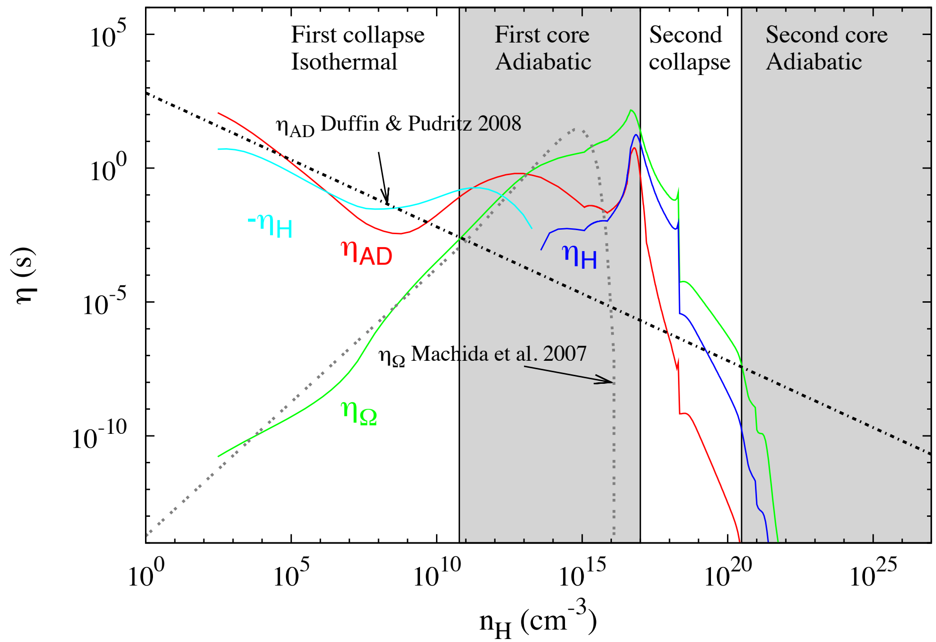
Ideal vs non-ideal MHD



Density (rendered) + Magnetic field lines
During first core phase.   Left: ideal MHD. Right: non-ideal MHD

Non-ideal MHD





$$\left.\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right|_{\mathrm{OR}} = -\boldsymbol{\nabla}\times\eta_{\mathrm{OR}}\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right),$$

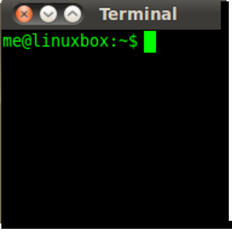
$$\left.\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right|_{\mathrm{HE}} = -\boldsymbol{\nabla}\times\eta_{\mathrm{HE}}\left[\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\times\hat{\boldsymbol{B}}\right],$$

$$\left.\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\right|_{\mathrm{AD}} = \boldsymbol{\nabla}\times\eta_{\mathrm{AD}}\left\{\left[\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\times\hat{\boldsymbol{B}}\right]\times\hat{\boldsymbol{B}}\right\}.$$

NICIL: Wurster (2016)

Marchand+ (2016)

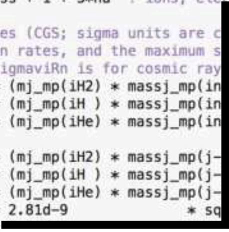
NICIL v1.2.3 is implemented in the current git version of Phantom



Non-ideal MHD in Phantom: the NICIL library

- *Phantom* includes the *NICIL* code (Wurster 2016)
    - Publically available at https://bitbucket.org/jameswurster/nicil
- When compiling, set NONIDEALMHD=yes
- Realistic defaults are set; these will self-consistently calculate the non-ideal coefficients
- Fully parameterisable
- Primary parameters are included in *Phantom*'s .in file
- All parameters are included at the top of nicil.F90
- Important parameters that can be modified
    - Included non-ideal MHD terms (default = ohmic + Hall + ambipolar)
    - Ionisation source (default = cosmic rays + thermal)
    - Cosmic ray ionisation rate (default = 10^{-17} s^{-1})
    - Elements that can be thermally ionised (cannot be modified through .in file)
    - Grain properties (default = fixed size of 0.1μm; alternate is MRN, but is slow)
- Important values are summarised in the dump files and the .ev file
- Can optionally preselect non-ideal MHD coefficients (preferably for tests only)

- All coefficients and required variables are calculated at runtime



Implementation

- Continuum equations

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} &= \left(\boldsymbol{B}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} - \boldsymbol{B}\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}\right) \\
&+ \boldsymbol{\nabla}\times\eta_{\text{art}}\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right) \\
&+ \boldsymbol{\nabla}\times\eta_{\text{OR}}\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right) \\
&+ \boldsymbol{\nabla}\times\eta_{\text{HE}}\left[\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\times\hat{\boldsymbol{B}}\right] \\
&+ \boldsymbol{\nabla}\times\eta_{\text{AD}}\left\{\left[\left(\boldsymbol{\nabla}\times\boldsymbol{B}\right)\times\hat{\boldsymbol{B}}\right]\times\hat{\boldsymbol{B}}\right\}
\end{aligned}
$$

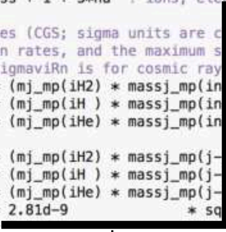
- SPMHD equations

$$
\frac{\mathrm{d}B_a^i}{\mathrm{d}t} = -\frac{1}{\Omega_a\rho_a}\sum_b m_b\left[v_{ab}^i B_a^j \nabla_a^j W_{ab}\left(h_a\right) - B_a^i v_{ab}^j \nabla_a^j W_{ab}\left(h_a\right)\right] + \left.\frac{\mathrm{d}B_a^i}{\mathrm{d}t}\right|_{\text{non-ideal}}
$$

$$
\left.\frac{\mathrm{d}\boldsymbol{B}_a}{\mathrm{d}t}\right|_{\text{non-ideal}} = -\rho_a\sum_b m_b\left[\frac{\boldsymbol{D}_a}{\Omega_a\rho_a^2}\times\nabla_a W_{ab}(h_a) + \frac{\boldsymbol{D}_b}{\Omega_b\rho_b^2}\times\nabla_a W_{ab}(h_b)\right],
$$

$$
\boldsymbol{D}_a^{\text{OR}} = -\eta_{\text{OR}}\boldsymbol{J}_a, \quad \boldsymbol{D}_a^{\text{HE}} = -\eta_{\text{HE}}\boldsymbol{J}_a\times\hat{\boldsymbol{B}}_a, \quad \boldsymbol{D}_a^{\text{AD}} = \eta_{\text{AD}}\left(\boldsymbol{J}_a\times\hat{\boldsymbol{B}}_a\right)\times\hat{\boldsymbol{B}}_a.
$$

Wurster, Price & Ayliffe (2014)



Implementation

Density Loop:

do i = 1, N

  do j = 1, N_{neigh}

    Using j, calculate density of i

    Using j, calculate current density, $\boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B}$, of i

  enddo

  Using new density of i, calculate η_{nimhd}

enddo

Force Loop:

do i = 1, N

  Calculate $\boldsymbol{J}_i \times \boldsymbol{B}_i$ and $(\boldsymbol{J}_i \times \boldsymbol{B}_i) \times \boldsymbol{B}_j$

  do j = 1, N_{neigh}

    Calculate $\boldsymbol{J}_j \times \boldsymbol{B}_j$ and $(\boldsymbol{J}_j \times \boldsymbol{B}_j) \times \boldsymbol{B}_i$

    Using j, calculate $\mathrm{d}\boldsymbol{B}/\mathrm{d}t_{\mathrm{non-ideal}}$ of i

  enddo

  Calculate non-ideal timesteps

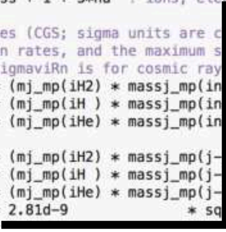
enddo

Step Loop:

do i = 1, N

  Updated magnetic field of i, using ideal, non-ideal and artificial terms

enddo

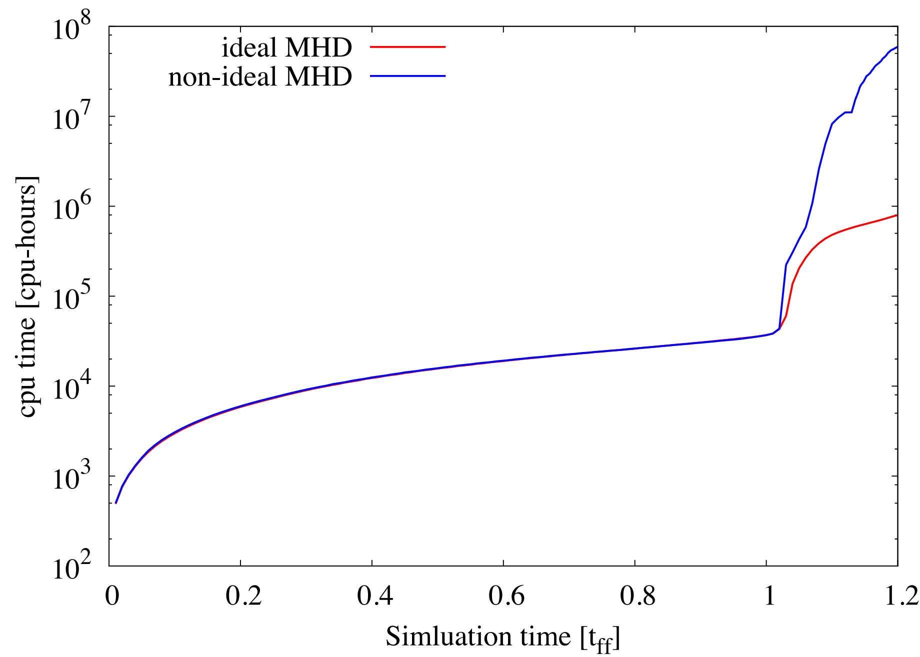


Implementation

- Timestepping:

$$\mathrm{d}t_{\mathrm{Courant}} = C_{\mathrm{c}} \frac{h}{v_{\mathrm{sig}}}$$

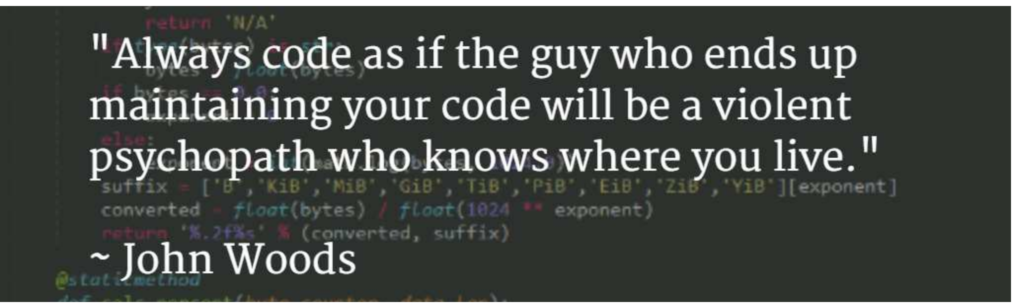
$$\mathrm{d}t_{\mathrm{nimhd}} = C_{\mathrm{ni}} \frac{h^2}{|\eta|}$$



- *Phantom* includes super-timestepping (Alexiades, Amiez & Gremaud 1996)
- Right: cpu-hours required for the 10^6 particle models with μ_0=5 in Wurster, Price & Bate (2016)
- Non-ideal MHD is slightly slower for $t < t_{\mathrm{ff}}$, and much slower for $t > t_{\mathrm{ff}}$

Conclusions

- Artificial resistivity is required to stabilised magnetohydrodynamics equations
- Ideal MHD is a poor approximation for modelling molecular clouds or protoplanetary discs
- Non-ideal MHD requires an assumption of chemistry
- The non-ideal MHD coefficients are not dependent on neighbours
- The non-ideal MHD contribution to the magnetic field evolution is dependent on neighbours
- Non-ideal MHD introduces a diffusion timestep $\propto h^2$, hence can be computationally expensive



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http://www.astro.ex.ac.uk/people/wurster/

Presentation available at http://www.astro.ex.ac.uk/people/wurster/files/spmhd_resistivity.pdf

Nicil's git repository: https://bitbucket.org/jameswurster/nicil