General Relativistic Hydrodynamics in Dynamical Spacetimes: A Particle Based Approach

Spencer Magnall,

Daniel Price, Paul Lasky

2023 Phantom Workshop, Monash University

MONASH University

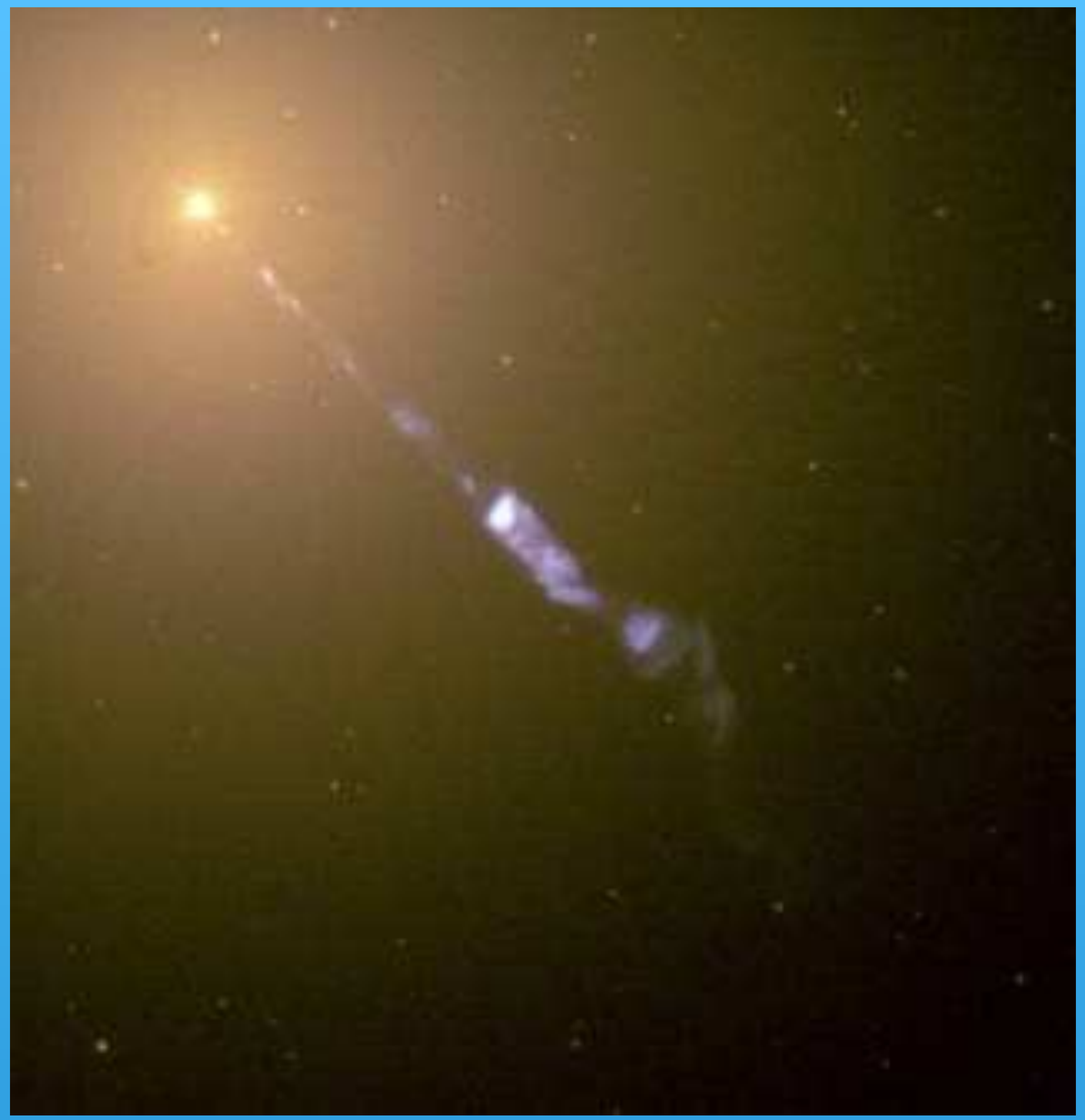
OzGrav

ARC Centre of Excellence for Gravitational Wave Discovery

Hydrodynamics in Astrophysics

- Many sources in astrophysics are fluids
- Complicated, generally don’t have analytical solutions
- How can we model them?
- **Computational fluid dynamics provides an answer!**

Relativistic Hydrodynamics



Credit: Nasa and the Hubble Heritage Team

- Fluid moving at relativistic speeds (e.g Jets)
- Fluid in the presence of a strong gravitational field (e.g Black Hole accretion)
- Solve the equations of relativistic Hydrodynamics for a given metric (eg. Kerr) →

$$\nabla_\mu T^{\mu\nu} = 0$$

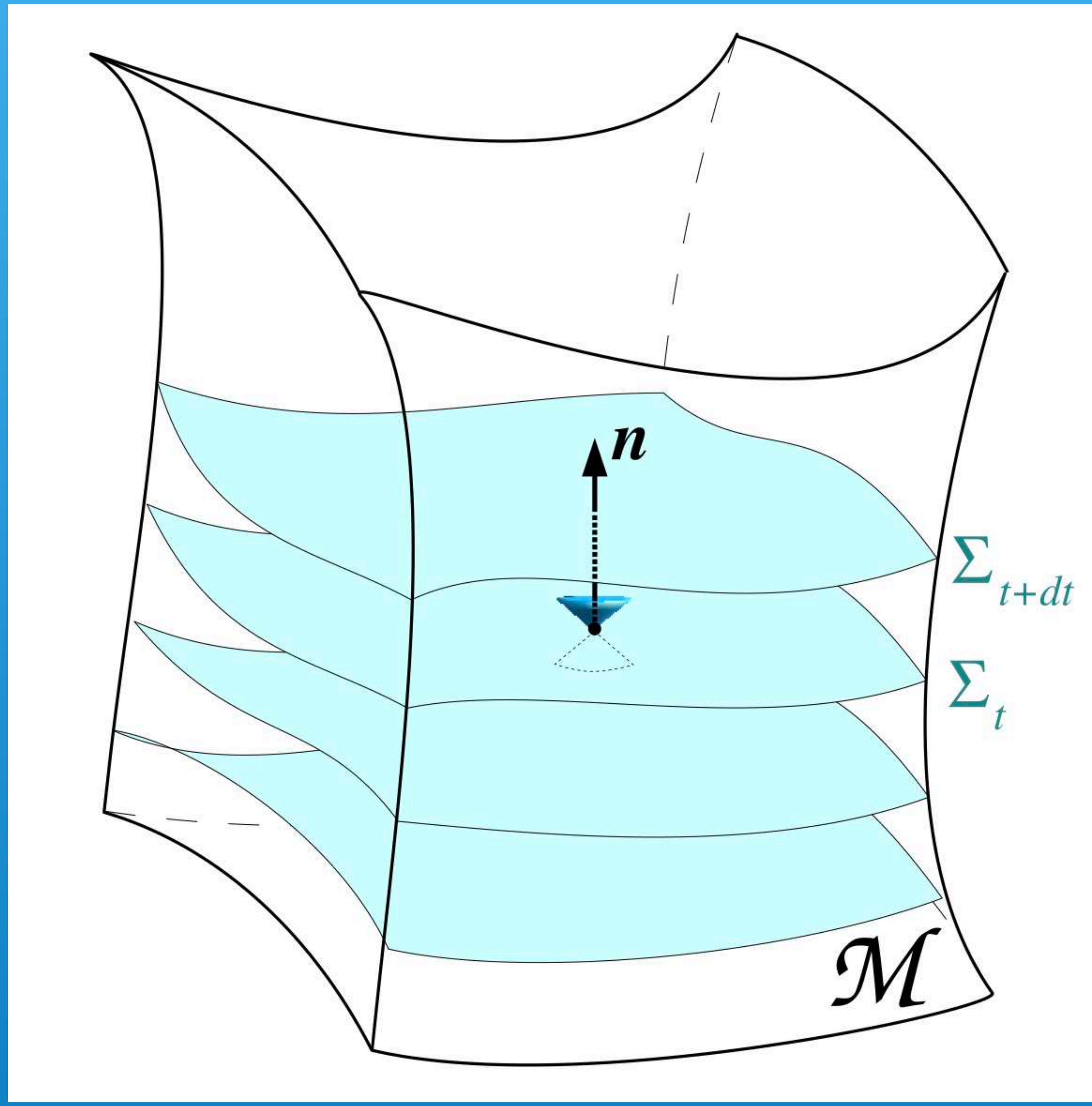
$$\nabla_\mu \rho u^\mu = 0$$

- **What about sources without analytical metric solutions?**

Numerical Relativity

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

- Based on splitting space-time to space 'slices'
- Solve one 'slice' of space at a time
- Widely used for the simulation of compact object mergers (eg. Binary Black Holes)

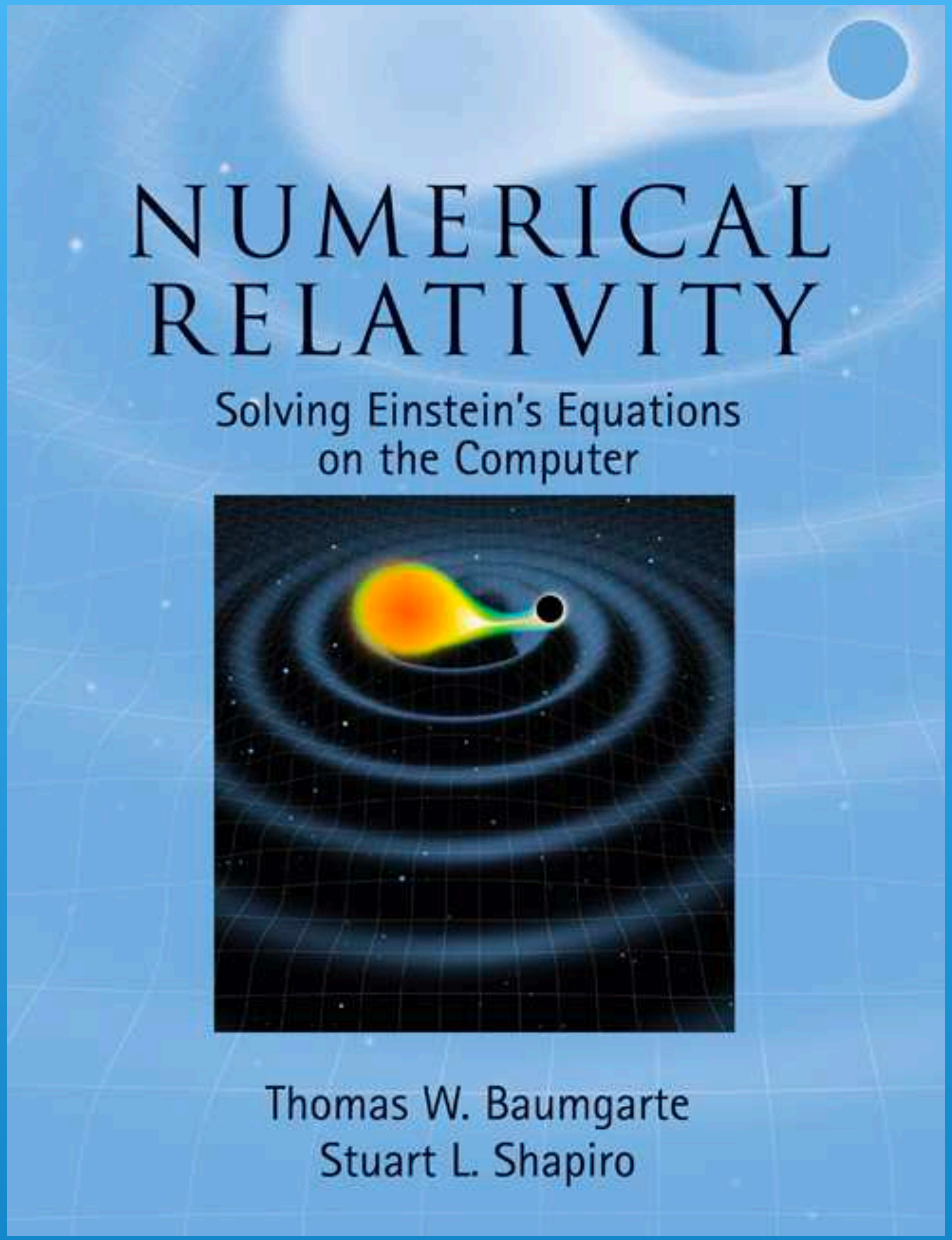


Gourgoulhon (2007)

spencer.magnall@monash.edu

Aside: NR is complicated…..

- Unfortunately don’t have time to discuss in depth
- I’ll try to keep the equations to a minimum..
- See textbook by Baumgarte and Shaprio for more



$$\partial_t \phi = -\frac{1}{2(D-1)}(\alpha K - \partial_i \beta^i) + \beta^i \partial_i \phi \,,$$

$$\partial_t K = -D_i D^i \alpha + \alpha\left(\tilde{A}^{ij}\tilde{A}_{ij} + \frac{K^2}{D-1}\right) + \frac{8\pi\alpha}{D-2}\Big((D-3)\rho + S\Big) + \beta^i \partial_i K \,,$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{jk}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{\gamma}_{ij} \,,$$

$$\begin{aligned}\partial_t \tilde{A}_{ij} = {} & e^{-4\phi}\{-D_i D_j \alpha + \alpha R_{ij} - 8\pi S_{ij}\alpha\}^{TF} + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) \\ & + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{D-1}\partial_k \beta^k \tilde{A}_{ij} \,,\end{aligned}$$

Relativistic Hydrodynamics + Numerical Relativity

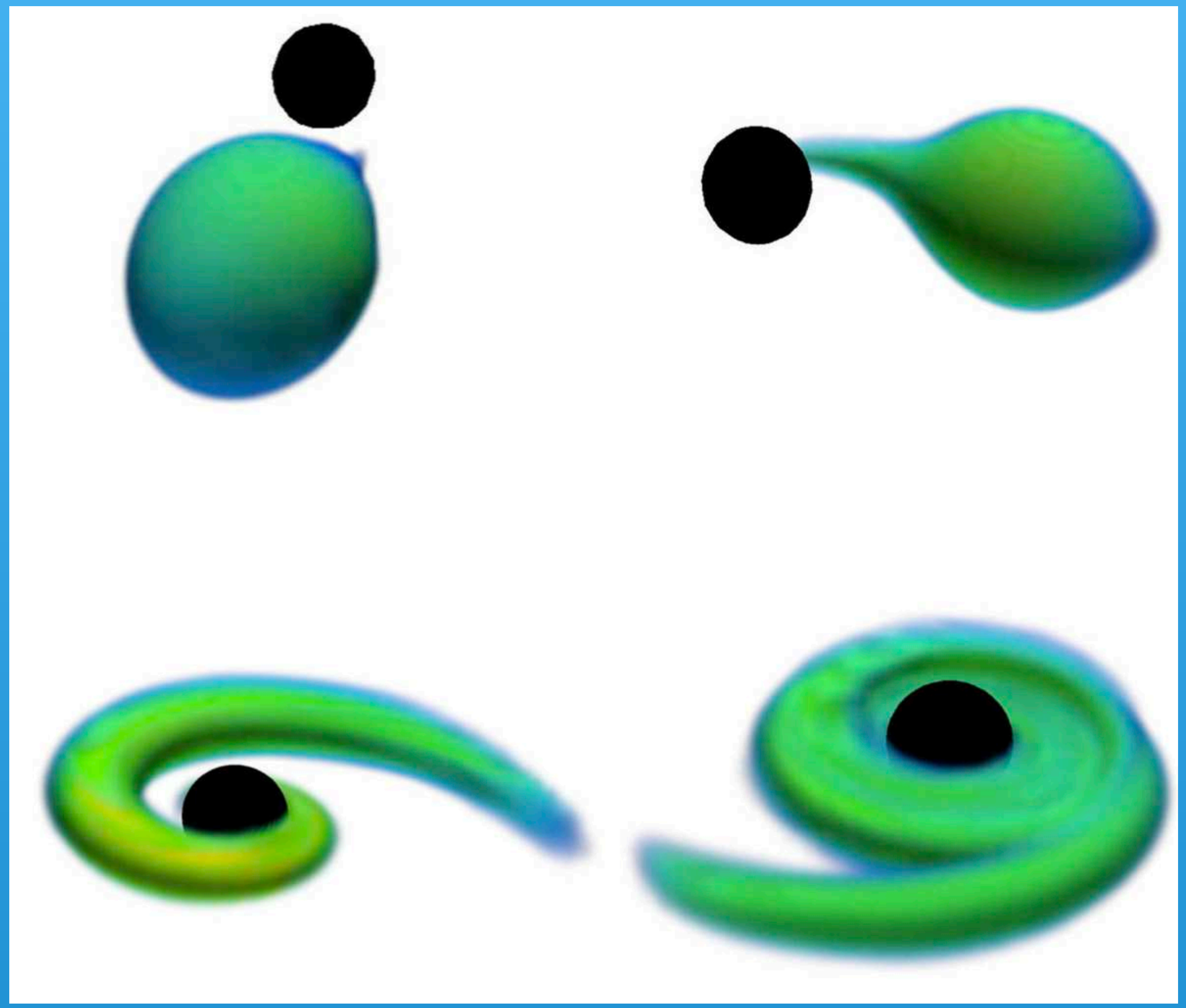
- Numerically solve the metric on the grid
- Solve the Hydrodynamics on the grid to calculate a stress energy tensor

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

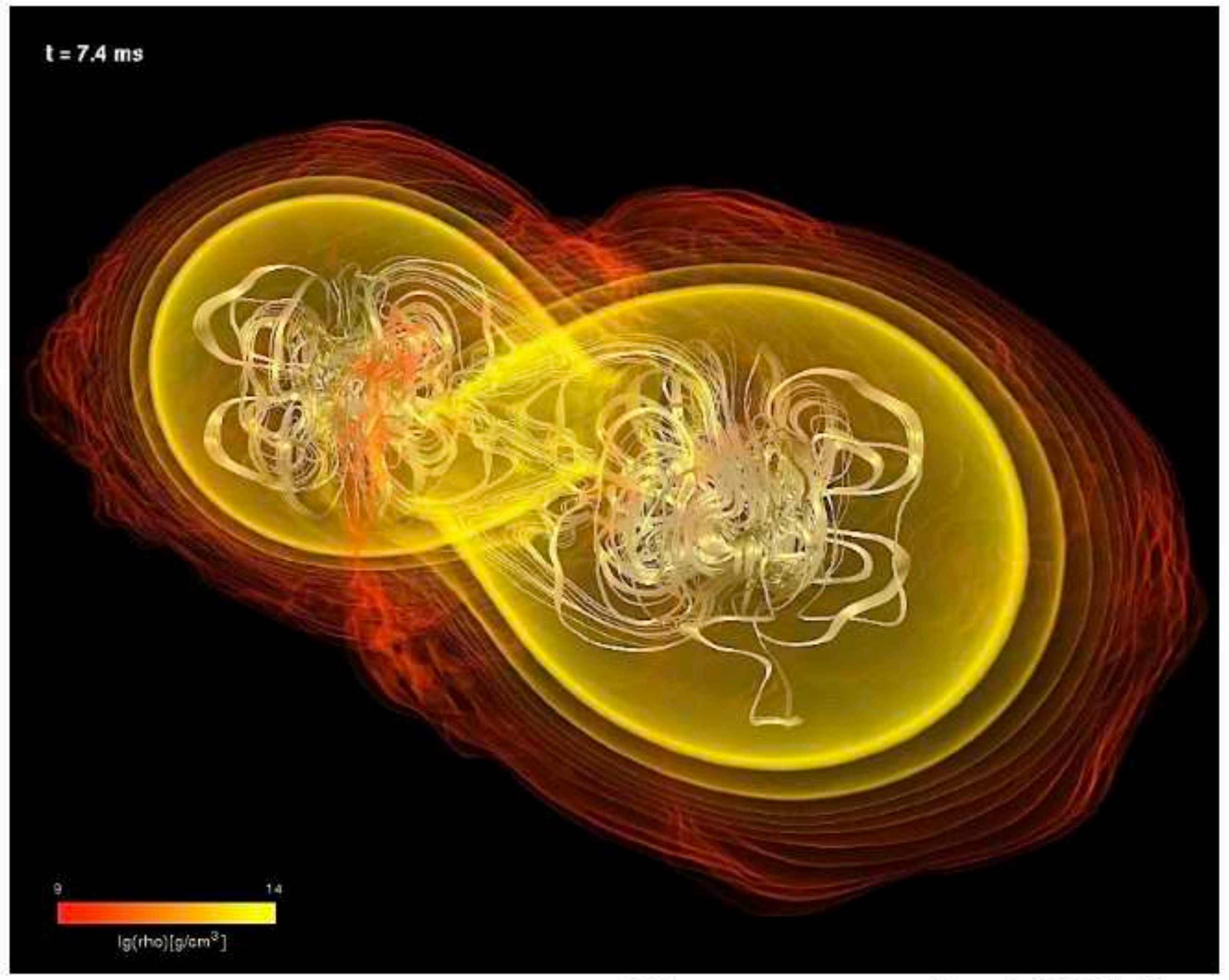
Numerical Relativity

Computational Fluid Dynamics

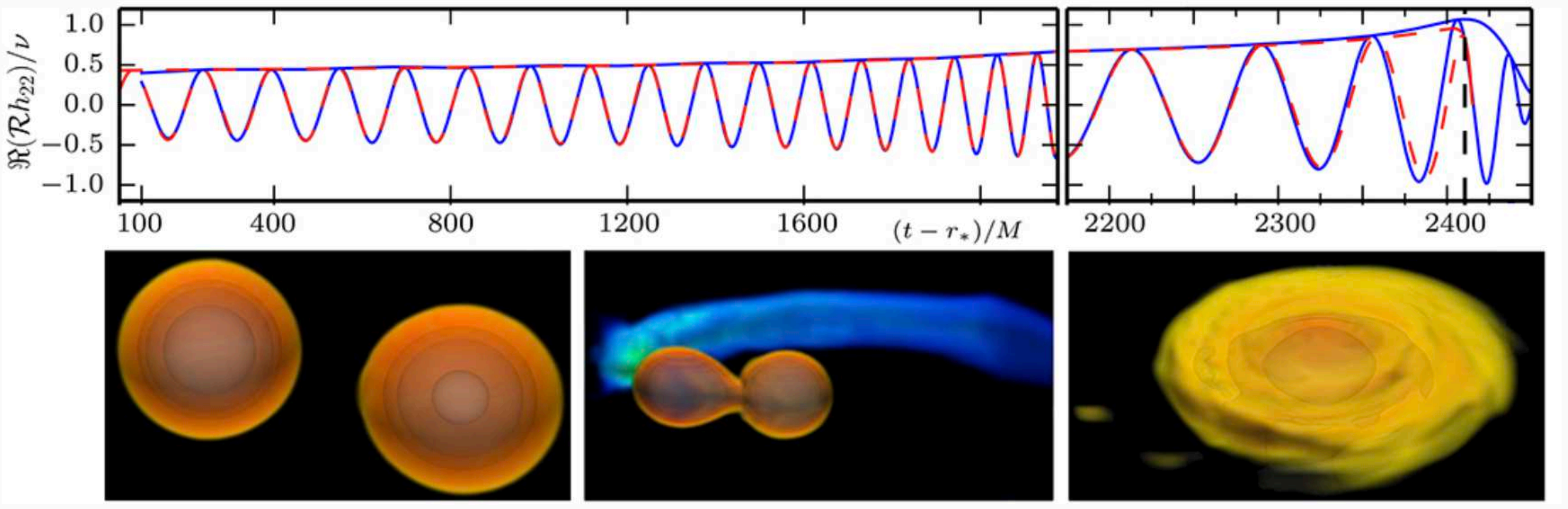
But Why?



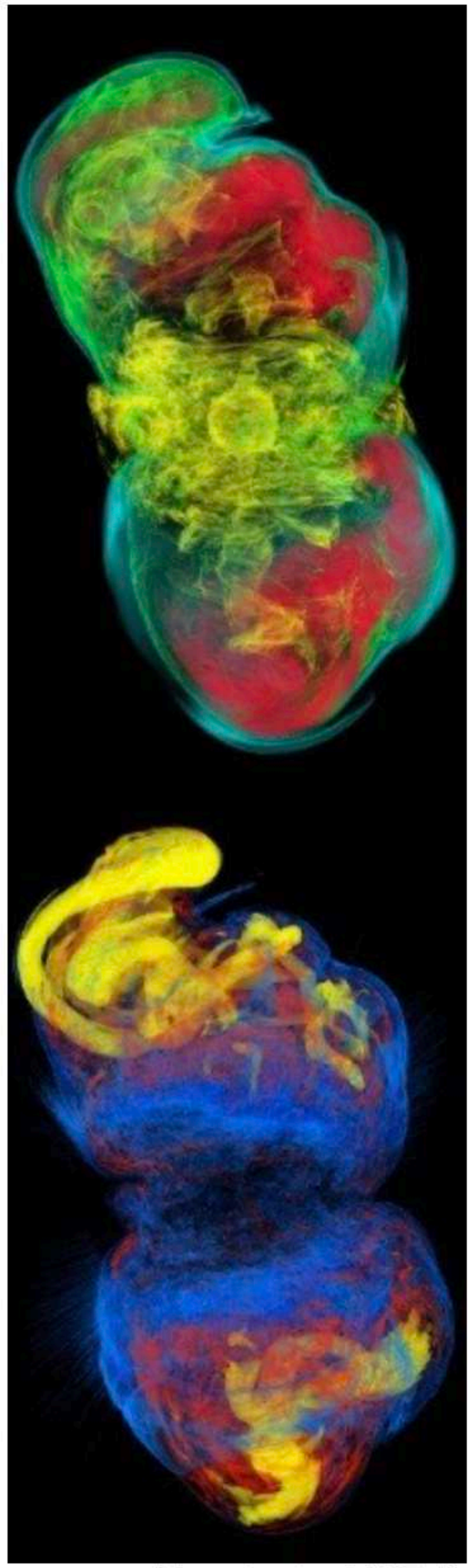
Foucart 2020



Giacomazzo et. al (2011)



Dietrich et al (2015)



Moesta et. al (2014)

What’s the Issue?

Downside to grid based methods

- Vacuum spacetimes with matter are not possible must use an atmosphere
- Neutron star surfaces?
- Difficulty tracking ejecta to large distances

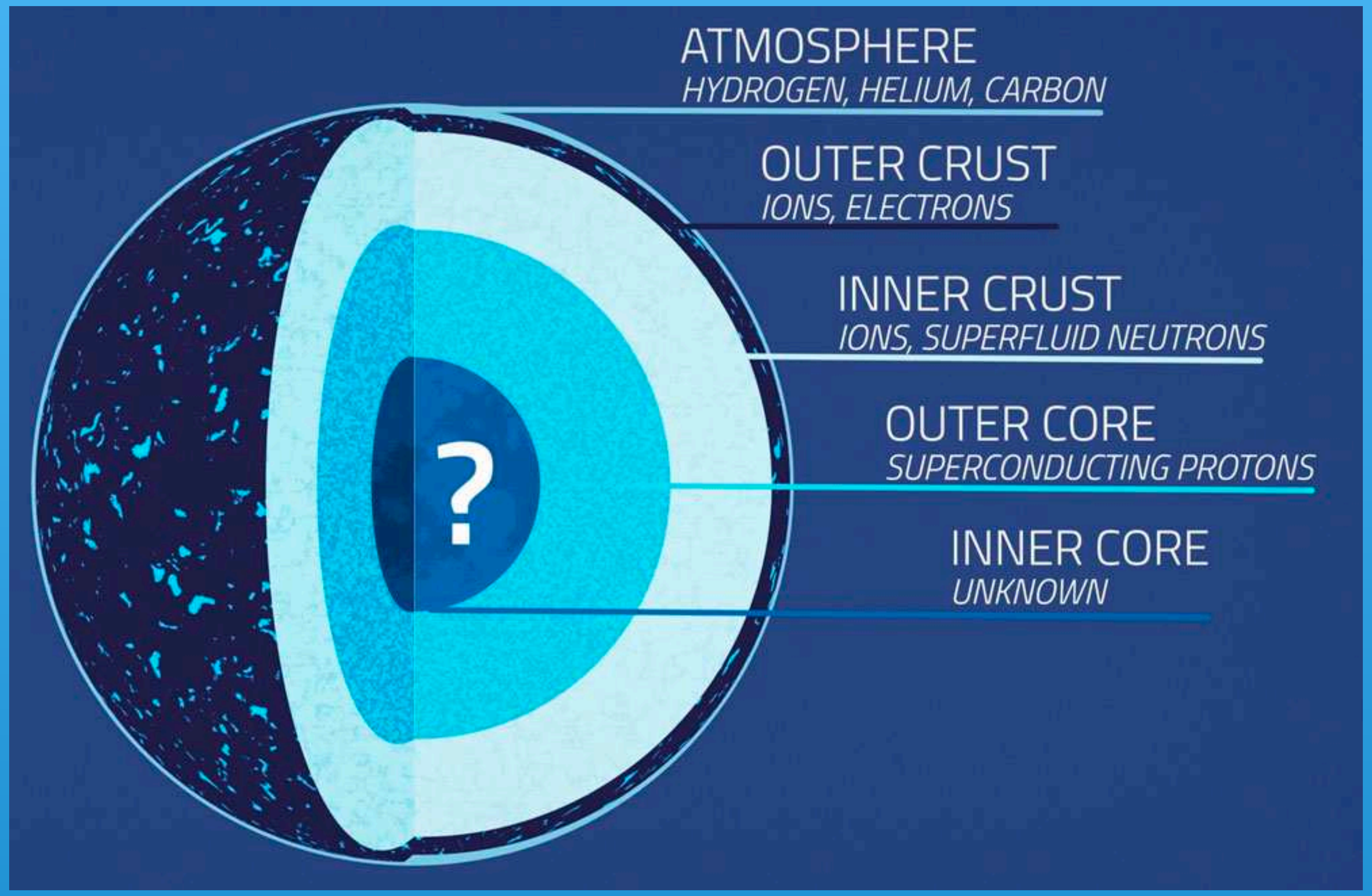


Image Credit: NASA’s Goddard Space Flight Center / Conceptual Image Lab

Inhomogenous Cosmology

- Simulations of the Universe including full GR
- Possible explanation of dark energy? (Buchert 2012)

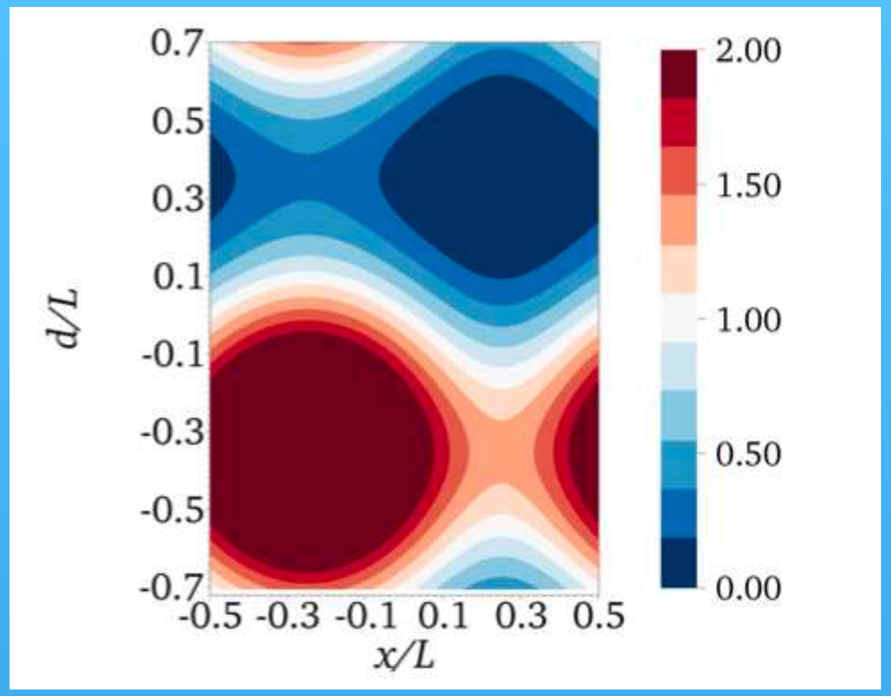
$$ds^2 = a(\eta)^2\left[-(1+2\Psi)d\eta^2 - 2B_i dx^i d\eta + (1-2\Phi)\delta_{ij}dx^i dx^j + h_{ij}dx^i dx^j\right]$$

Newtonian Potential

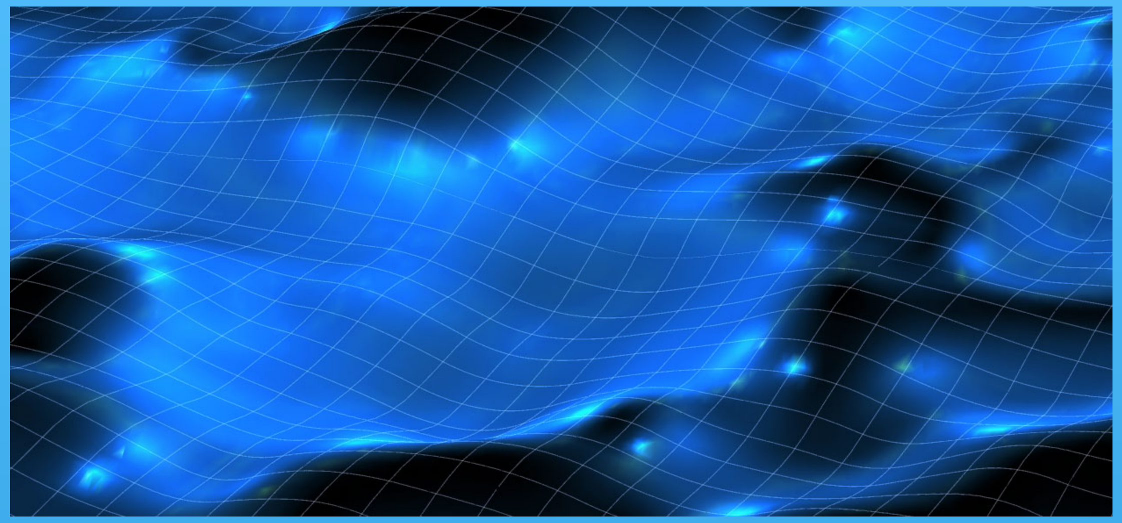
Relativistic Pertubations

Spencer Magnall, Monash University

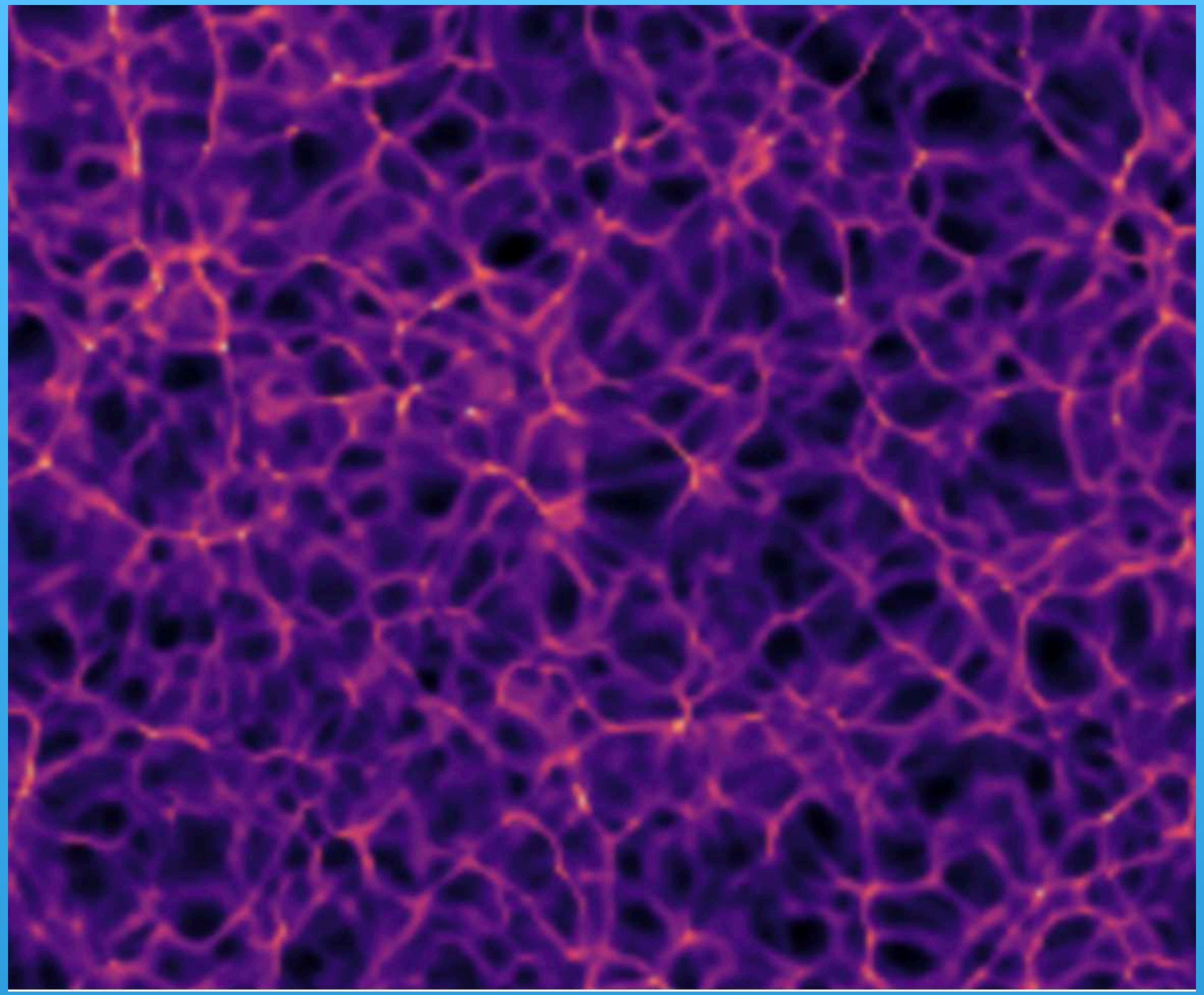
spencer.magnall@monash.edu



Bentivegna & Bruni (2016) Bentivegna (2016)



Mertens et. al (2016),
Giblin et. al (2016-2019)

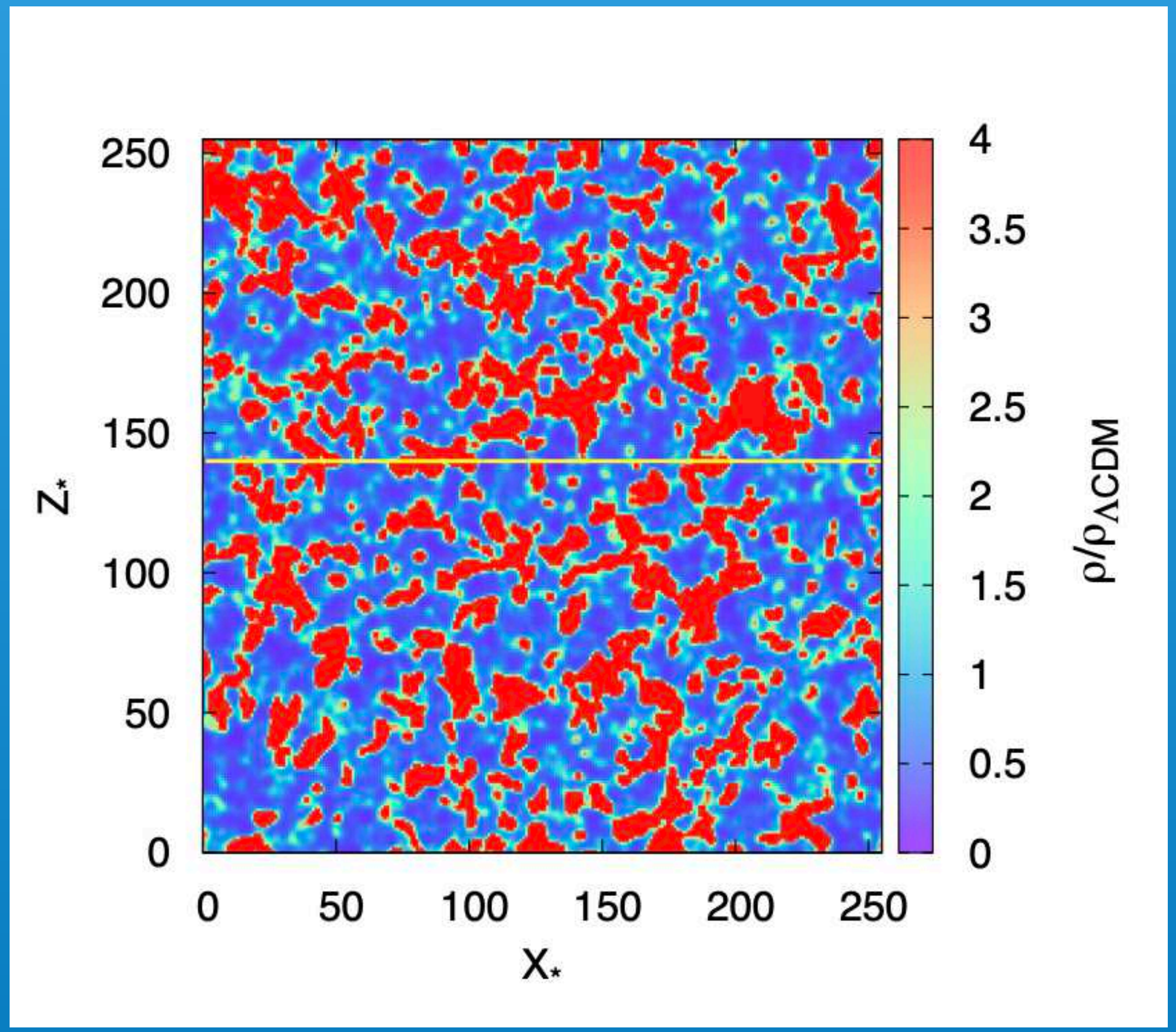


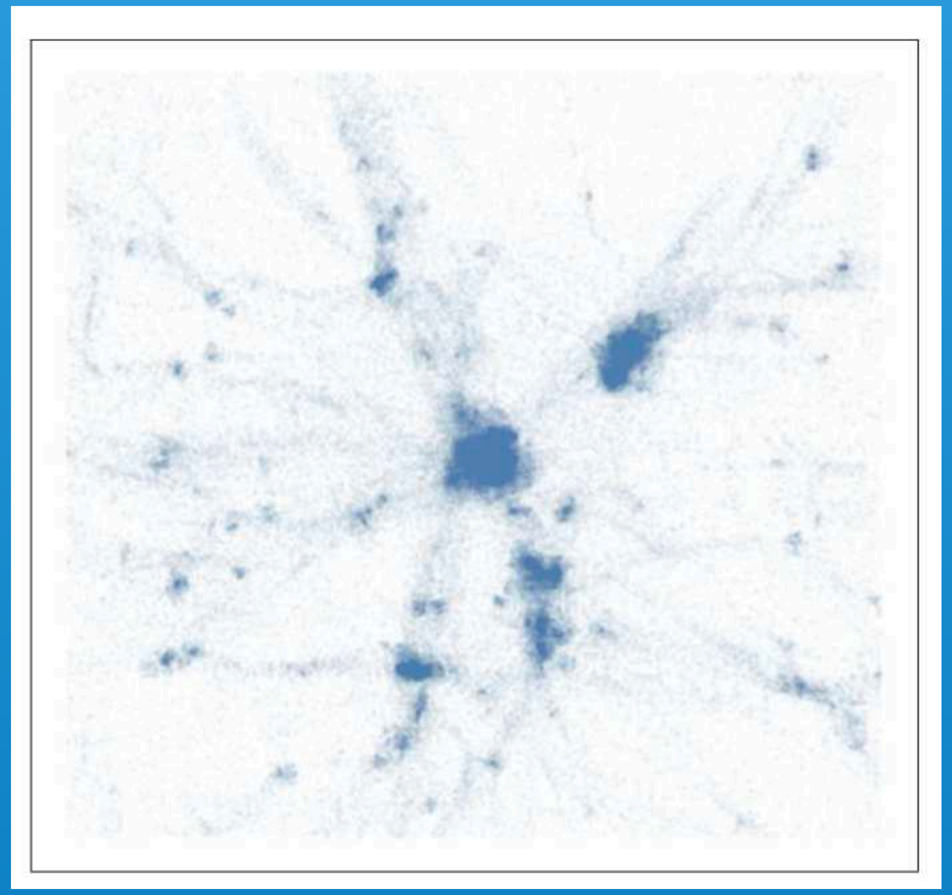
Daverio et. al (2017,2019),
East et. al (2018)

Boljeko (2017)

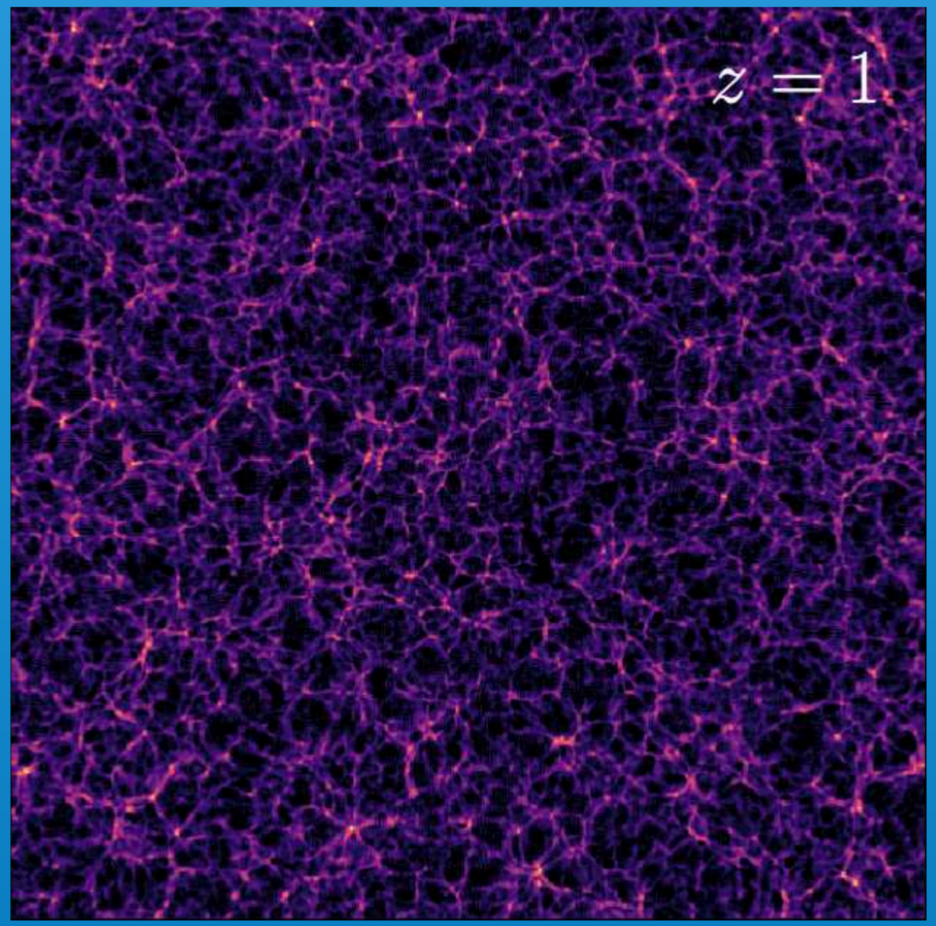
Some approximations to GR

Macpherson et. al 2017; 2019





Adamek et al. 2016



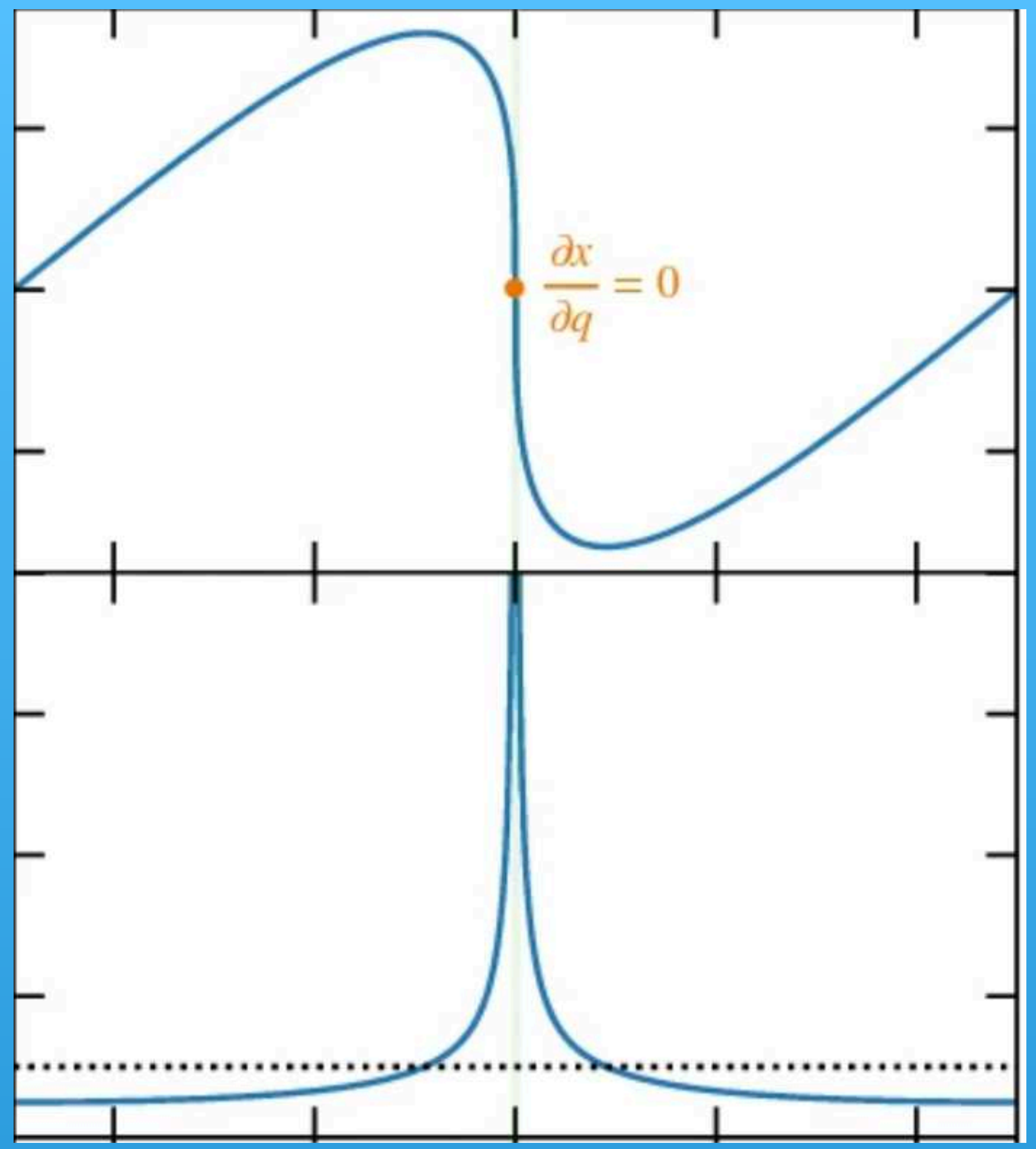
Barrera-Hinojosa et al. 2020

Shell Crossings

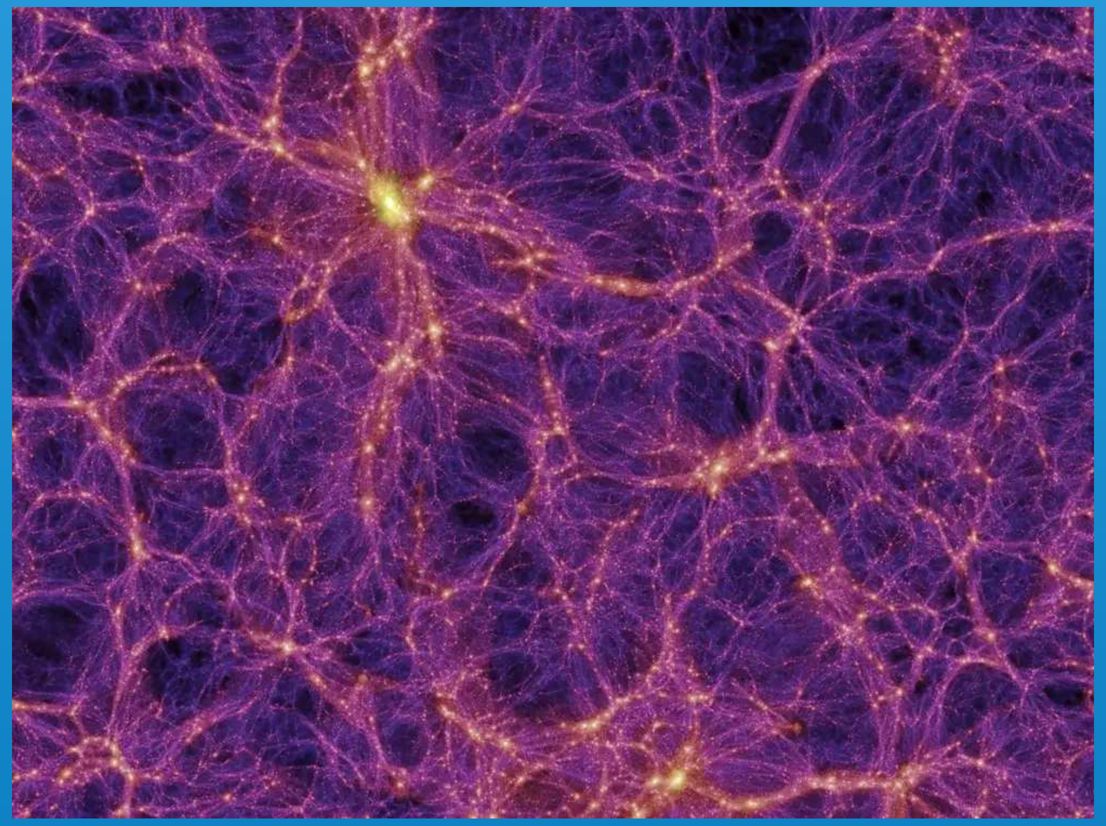
Velocity Becomes Double Valued

$\rho \to \infty$

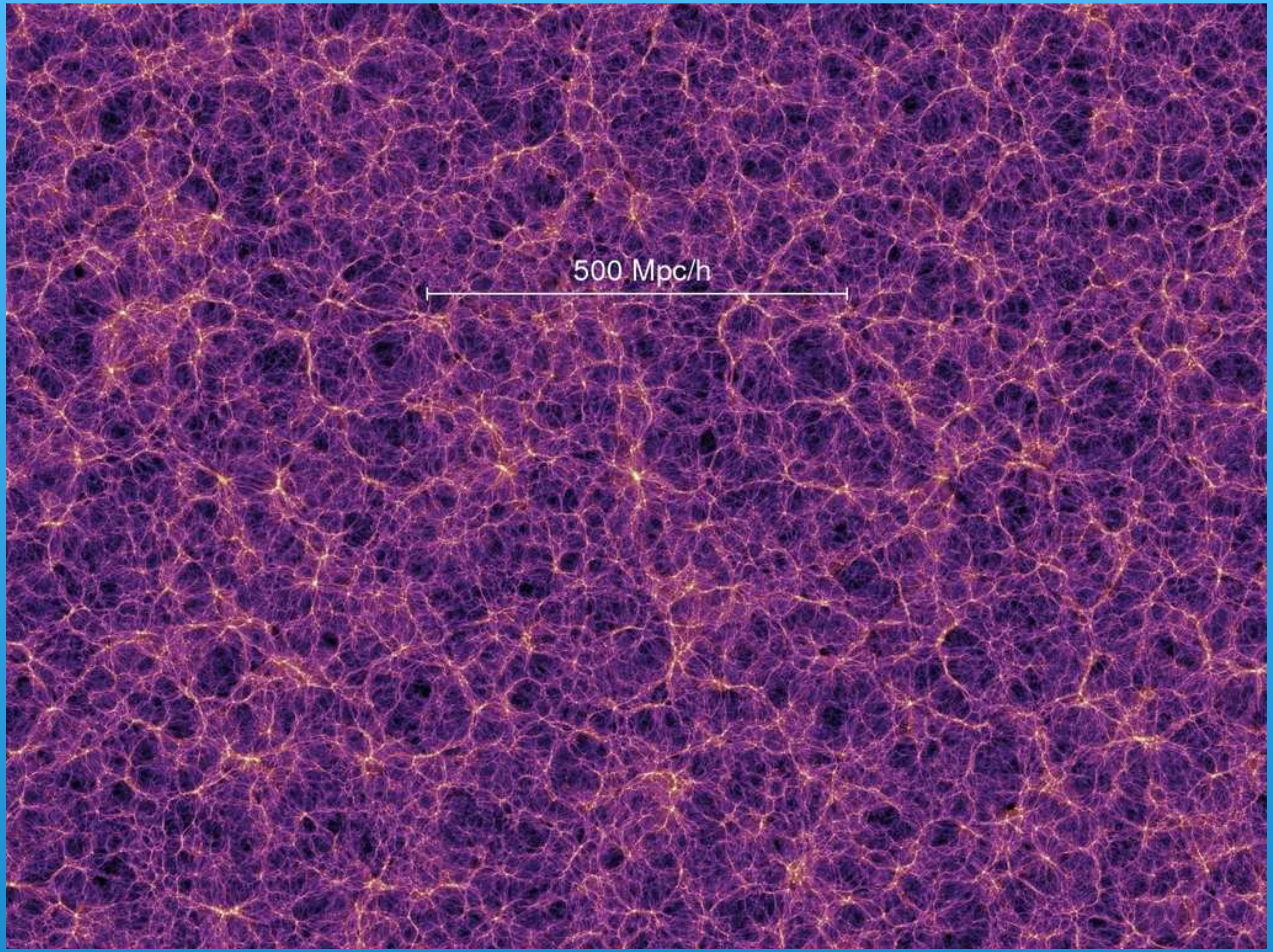
- Most Simulations of Cosmology with NR are unable to handle shell crossing without special treatment
- Formation of Dark Matter "Halos" like traditional N-Body codes is not possible



Angulo and Hahn (2022)

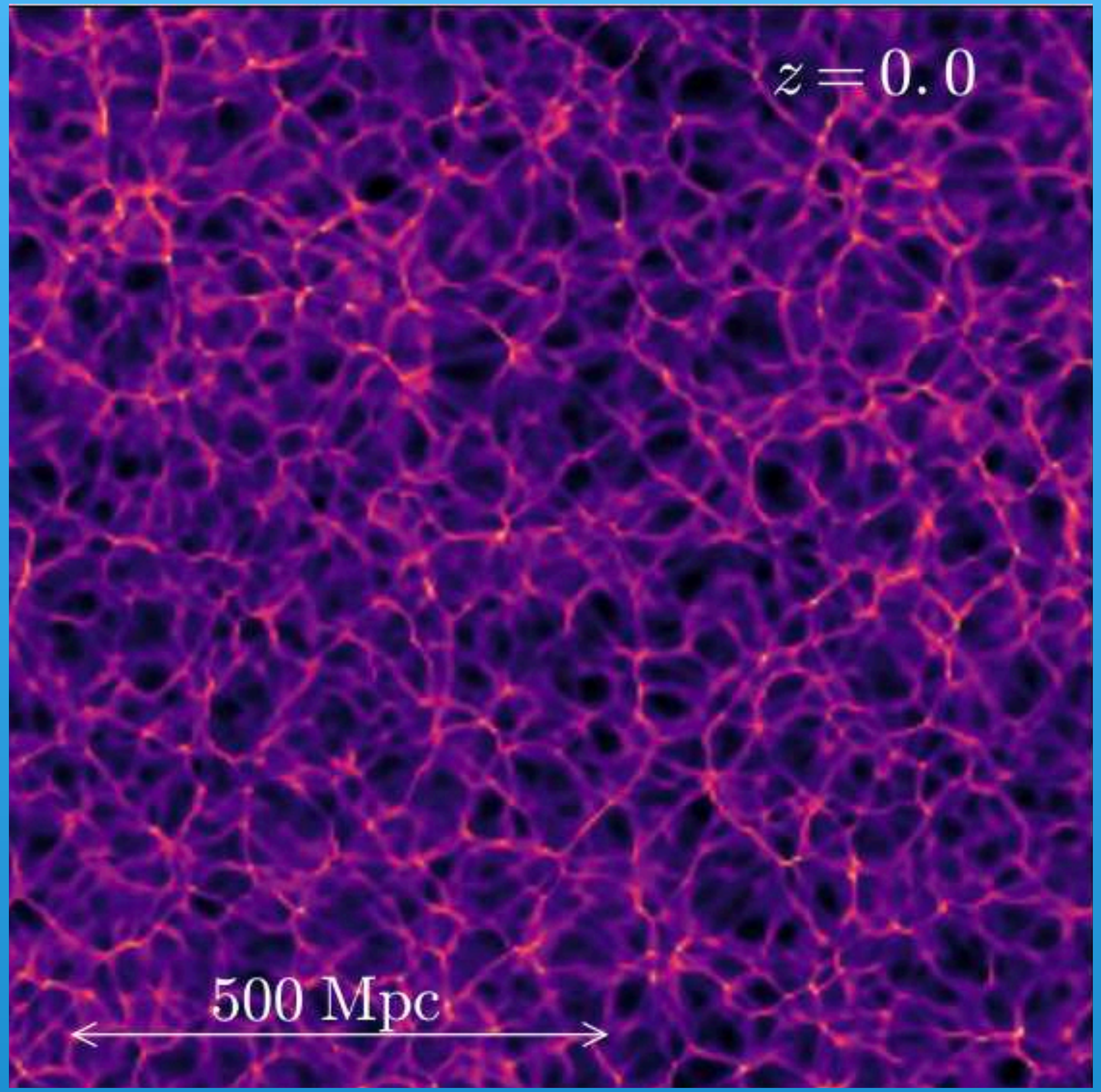


Springel et. al 2005



Springel et. al 2005

Newtonian N-Body



Macpherson et. al 2019

Numerical Relativity

The Solution?

- **Use Particles for the stress energy tensor evolution!**
- **Use Lagrangian Particle Based Hydrodynamics**
- Evolve the particles then calculate a new calculate a new stress energy tensor

Lagrangian: Move with the fluid, like a boat on a river

Einstein Toolkit

- Publicly Available, Widely Used, Well Tested framework for Numerical Relativity
- Solves the Field Equations using the BSSN (Shibata and Nakamura, 1995; Baumgarte and Shapiro, 1999) formalism
- Able to Initialise spacetimes for Binary Black Holes, Binary Neutron Stars, and Neutron Star-Black hole
- **Initial Conditions for Cosmology provided by FLRWSolver (Macpherson+ 2017)**

einstein toolkit

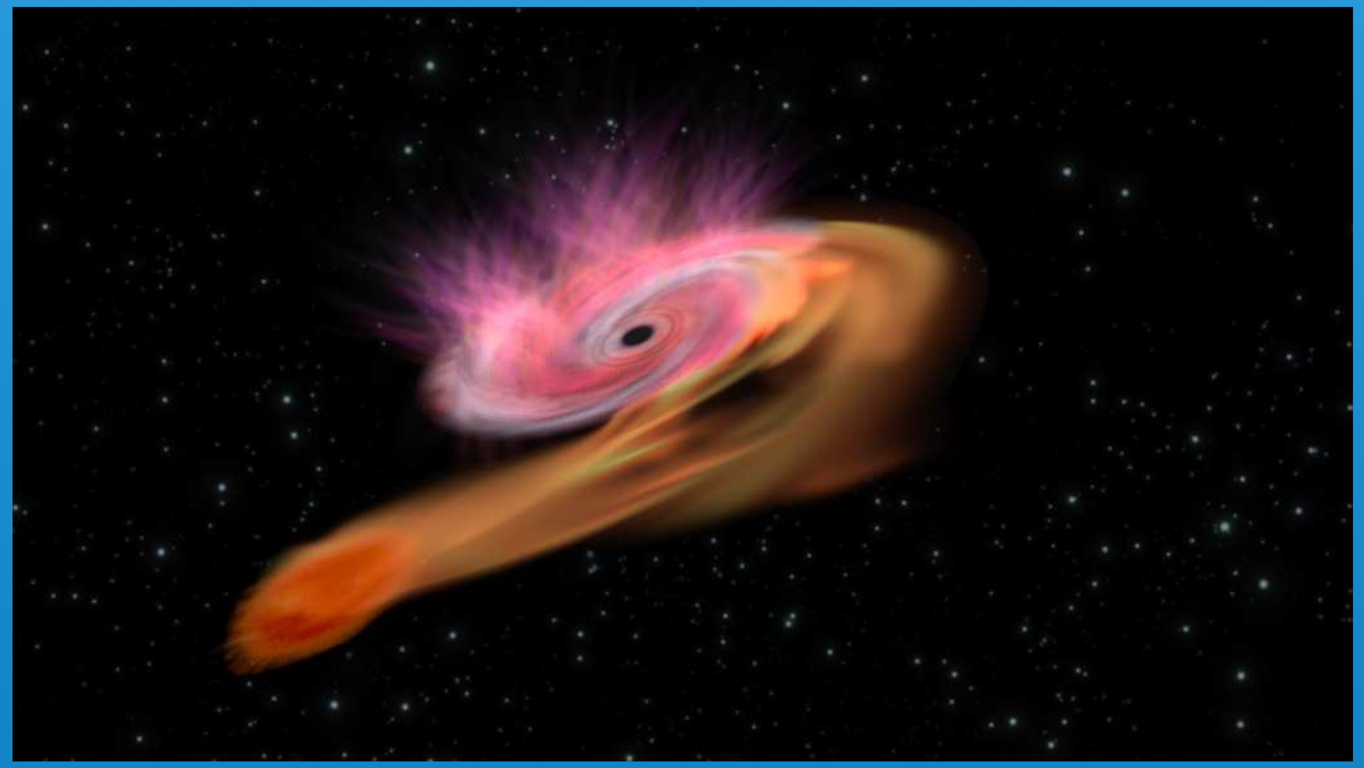
Löffler et al (2012)

Phantom

- Publicly available Smoothed Particle Hydrodynamics code
- Lagrangian mesh-free particle based method
- Supports General Relativistic Hydrodynamics for any fixed metric (Liptai and Price 2019)

Darbha et. al 2021

Neutron Star-Black Hole Kilonovae
(Image Credit: Carl Knox)



Toscani et. al 2022

Tidal Disruption Events
(Image Credit: ESA/C. Carreau)

See also talks by Martina, Megha and Fitz!

einstein toolkit

Metric Evolution

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Hydrodynamic Evolution

PHANTOM

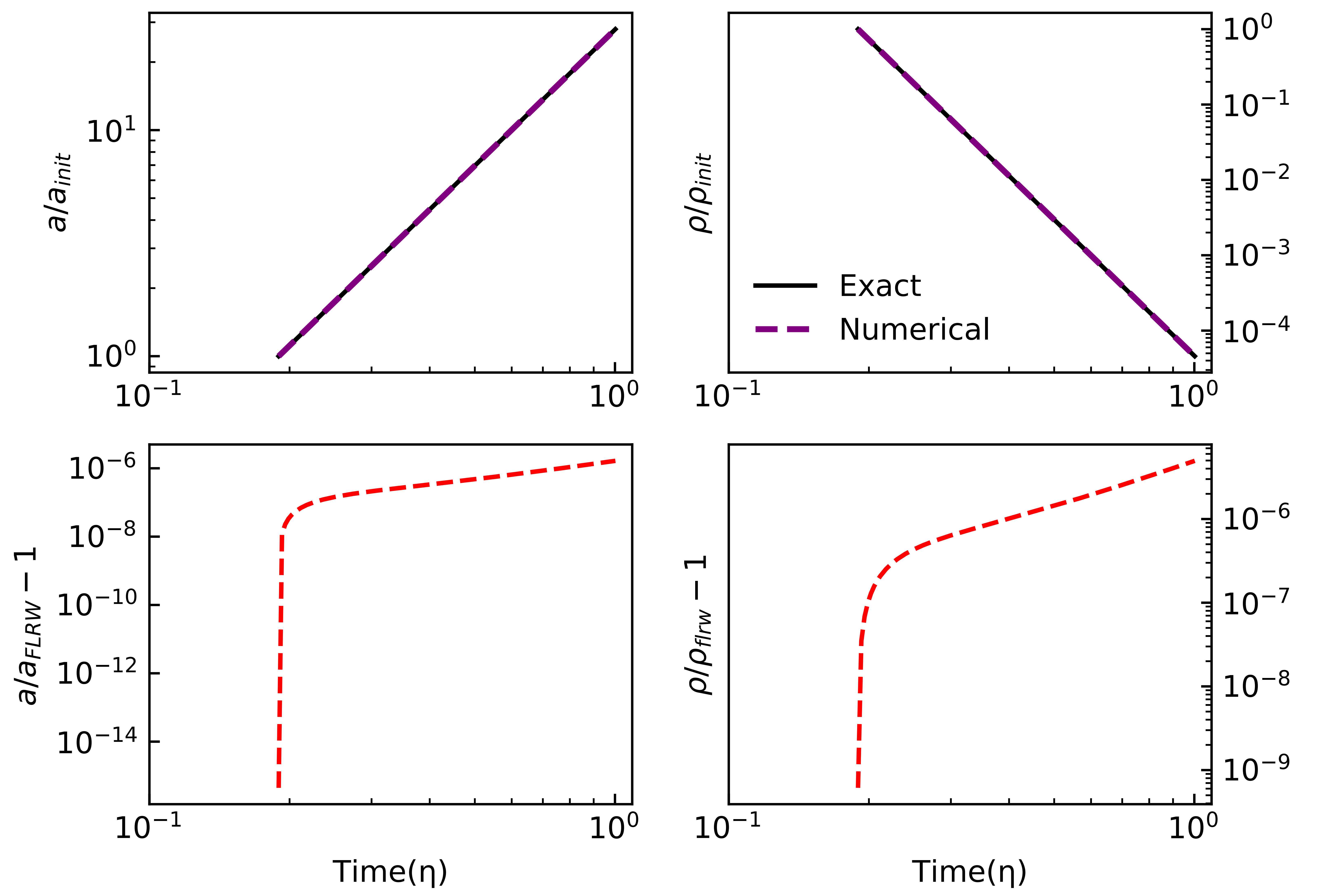
Exact interpolation of Petkova et. al 2018

$g^{n}_{\mu\nu}$ → Interpolate from Grid to Particles → Evolve Particles: Calculate Stress Energy Tensor → $T^{Particles}_{\mu\nu}$ → Kernel Interpolation to Grid → $T^{Grid}_{\mu\nu}$ → Evolve the metric → $g^{n+1}_{\mu\nu}$

Does It Work?

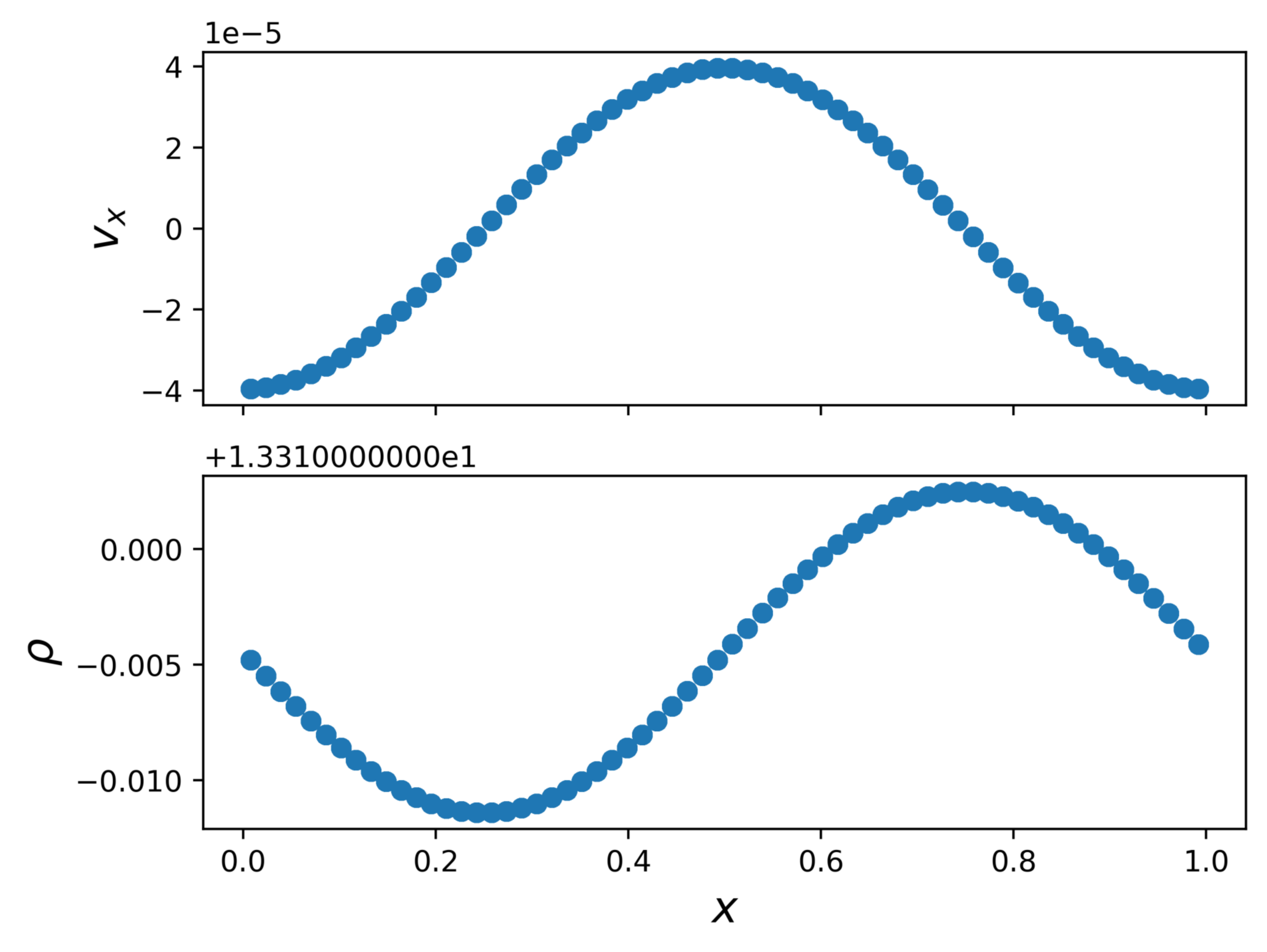
Consider a matter dominated FLRW universe:

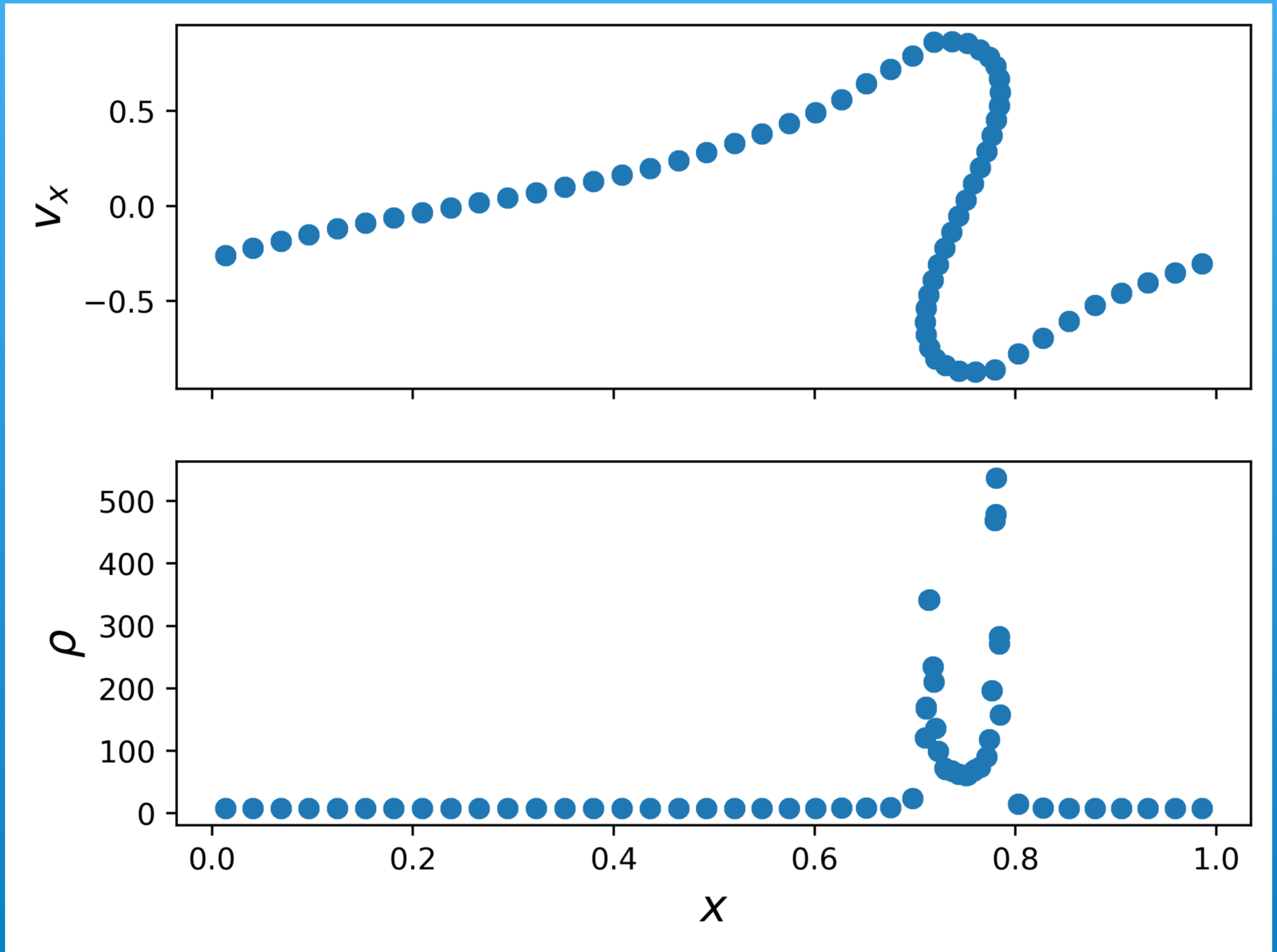
- Periodic Boundary Conditions
- Constant Density, Pressure-less fluid
- No dark energy, $\Lambda = 0$ (Einstein de Sitter Universe)
- Flat $k = 0$



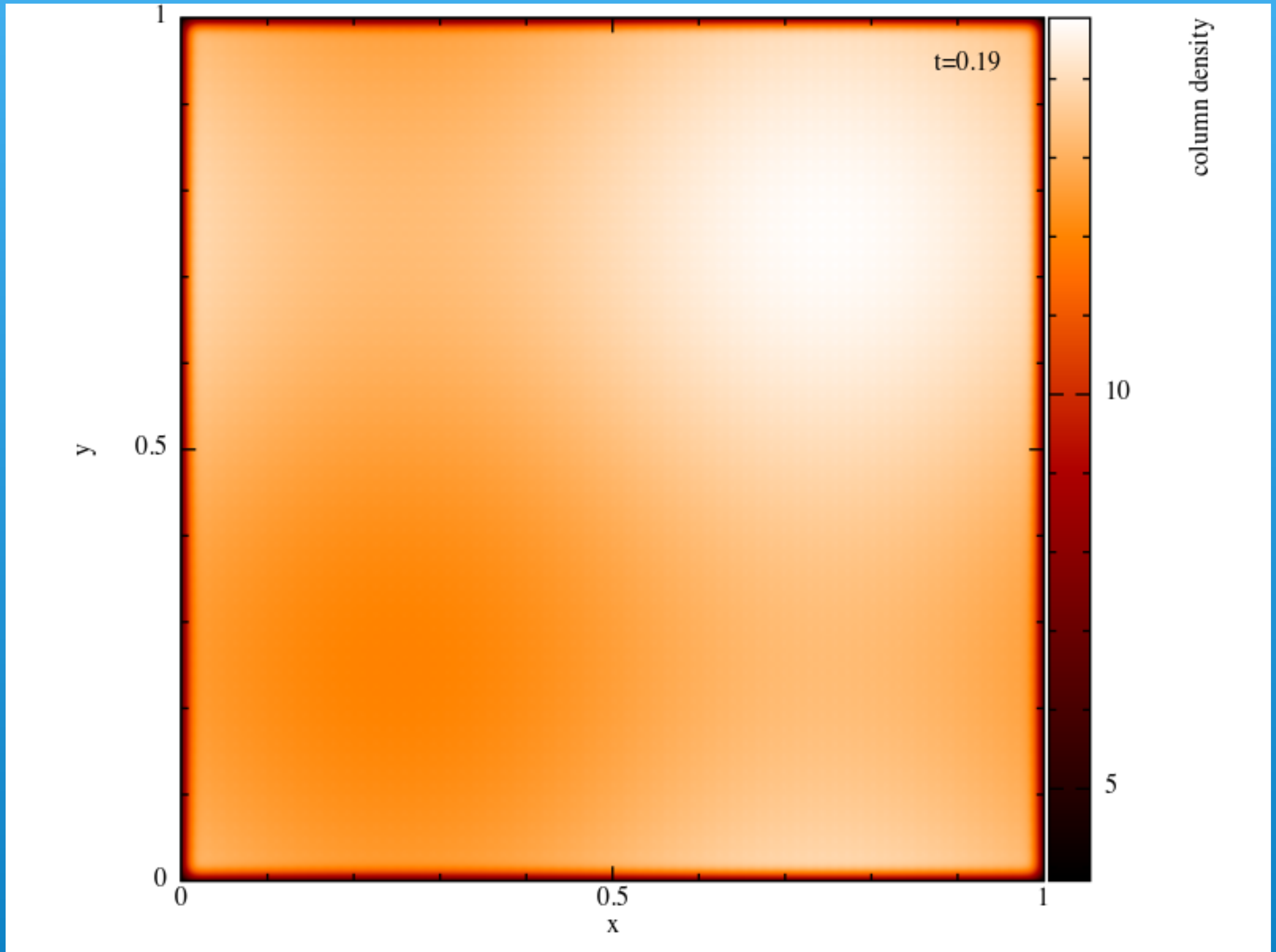
a/a_{init}
10^1
10^0
10^{-1}
10^0
ρ/ρ_{init}
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Exact
Numerical
$a/a_{FLRW} - 1$
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
$\rho/\rho_{flrw} - 1$
10^{-6}
10^{-7}
10^{-8}
10^{-9}
Time(η)
Time(η)

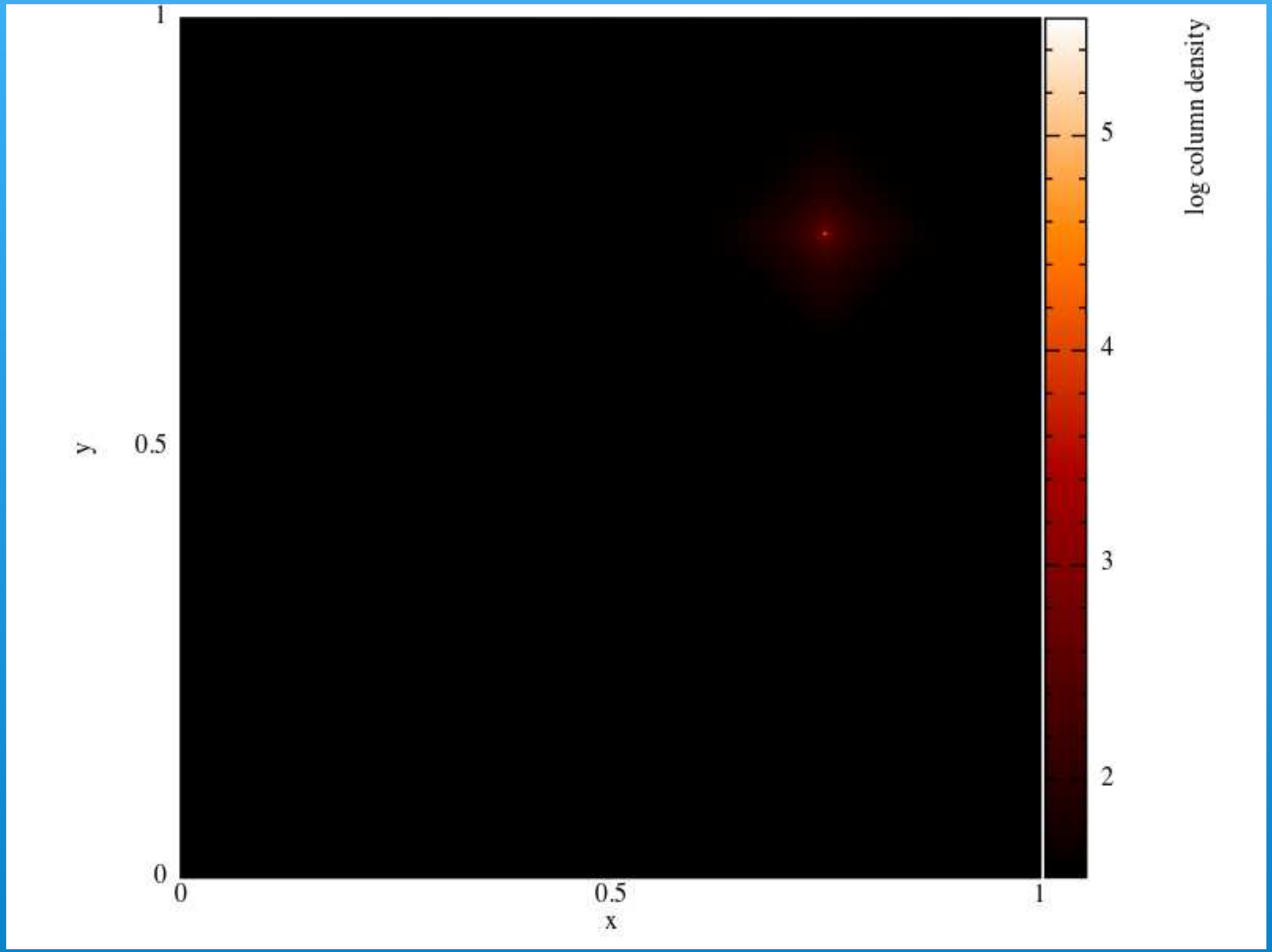
Shell Crossings





3D Collapse


t=0.19
column density
10
5
y
0.5
1
0
0.5
1
x


log column density
5
4
3
2
y
0.5
1
0
0.5
1
x

Summary

- Particle Based Hydrodynamics provides a solution for shell-crossing singularities in NR!
- Realistic structure formation using NR is possible.
- The method can be easily extended to the simulation of compact binary mergers such as Binary Neutron Star Mergers or Neutron Star - Black Hole

Gauge Conditions

$$\partial_t \alpha = -f(\alpha)\alpha^2 K$$

$$f = 1/3$$

$$\partial_t \alpha = \partial_t a$$

$$\beta^i = 0$$

$$t = \eta$$

Equations of Hydrodynamics

$$\frac{d\rho^*}{dt} = -\rho^* \frac{\partial v^i}{dx^i}$$

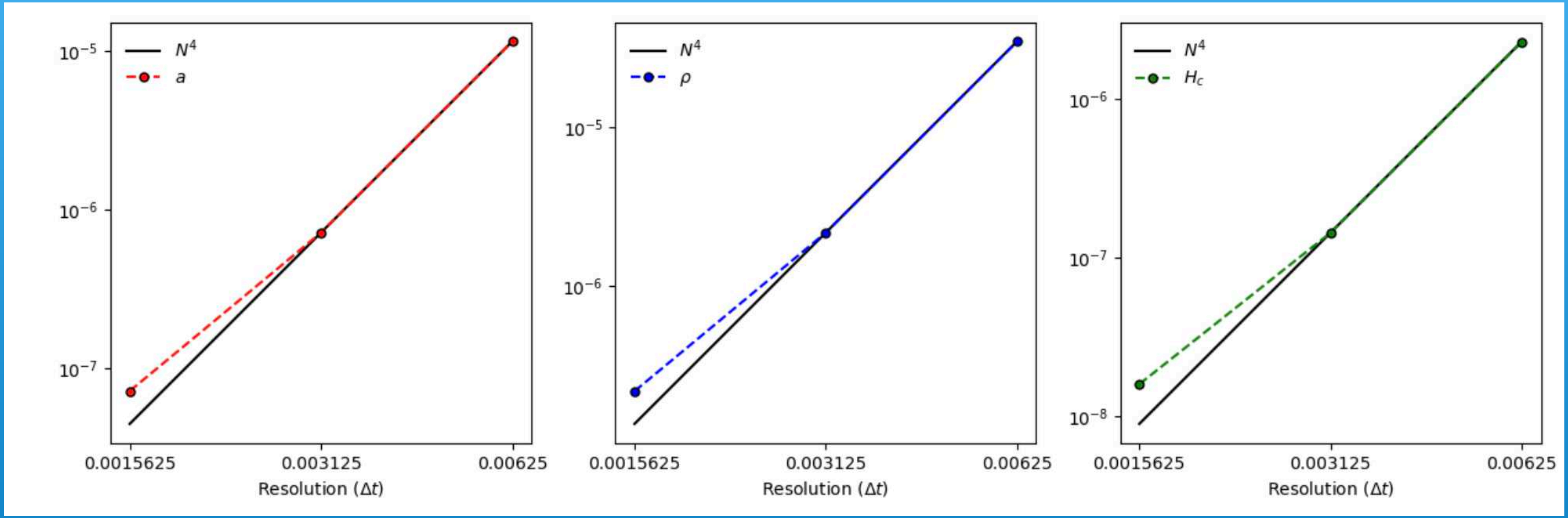
$$\frac{dp^i}{dt} = -\frac{1}{\rho^*}\frac{\partial(\sqrt{-g}P)}{dx^i} + f_i$$

$$\frac{de}{dt} = -\frac{1}{\rho^*}\frac{\partial(\sqrt{-g}Pv^i)}{dx^i} + \Lambda_i$$

$$f_i = \frac{\sqrt{-g}}{2\rho^*}\left(T_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^i}\right)$$

$$\Lambda_i = -\frac{\sqrt{-g}}{2\rho^*}\left(T_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial t}\right)$$

Numerical Convergence



Rosswog & Diener

- Similar method using SPH and NR Rosswog et. al 2022
- Applied to the simulation of Binary neutron star mergers in Diener et. al 2022