Semi-Analytic Models of Spiral Planet Wakes

+ some applications

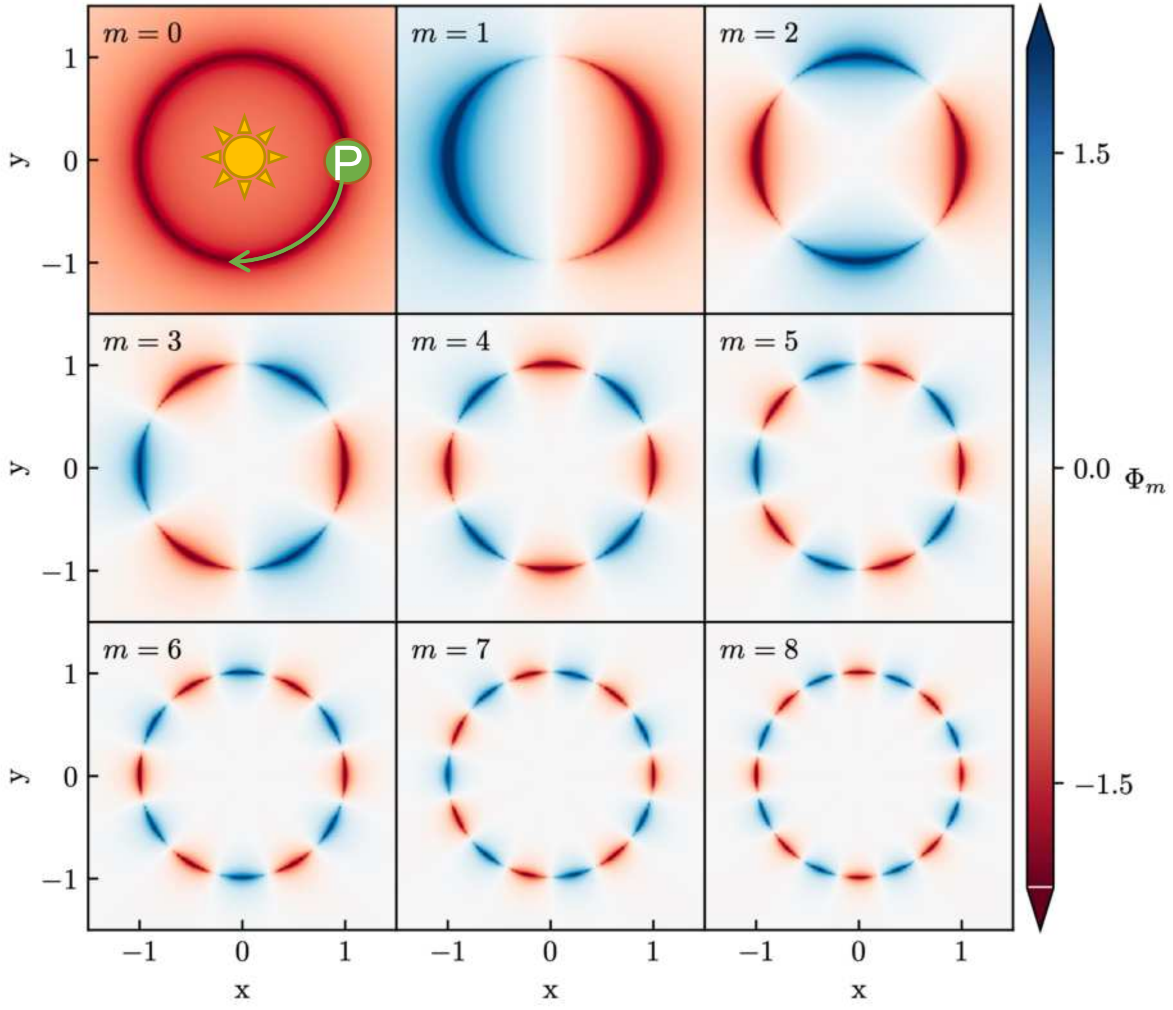
Thomas Hilder

Background image: wakeflow.readthedocs.io

Why do we get this?

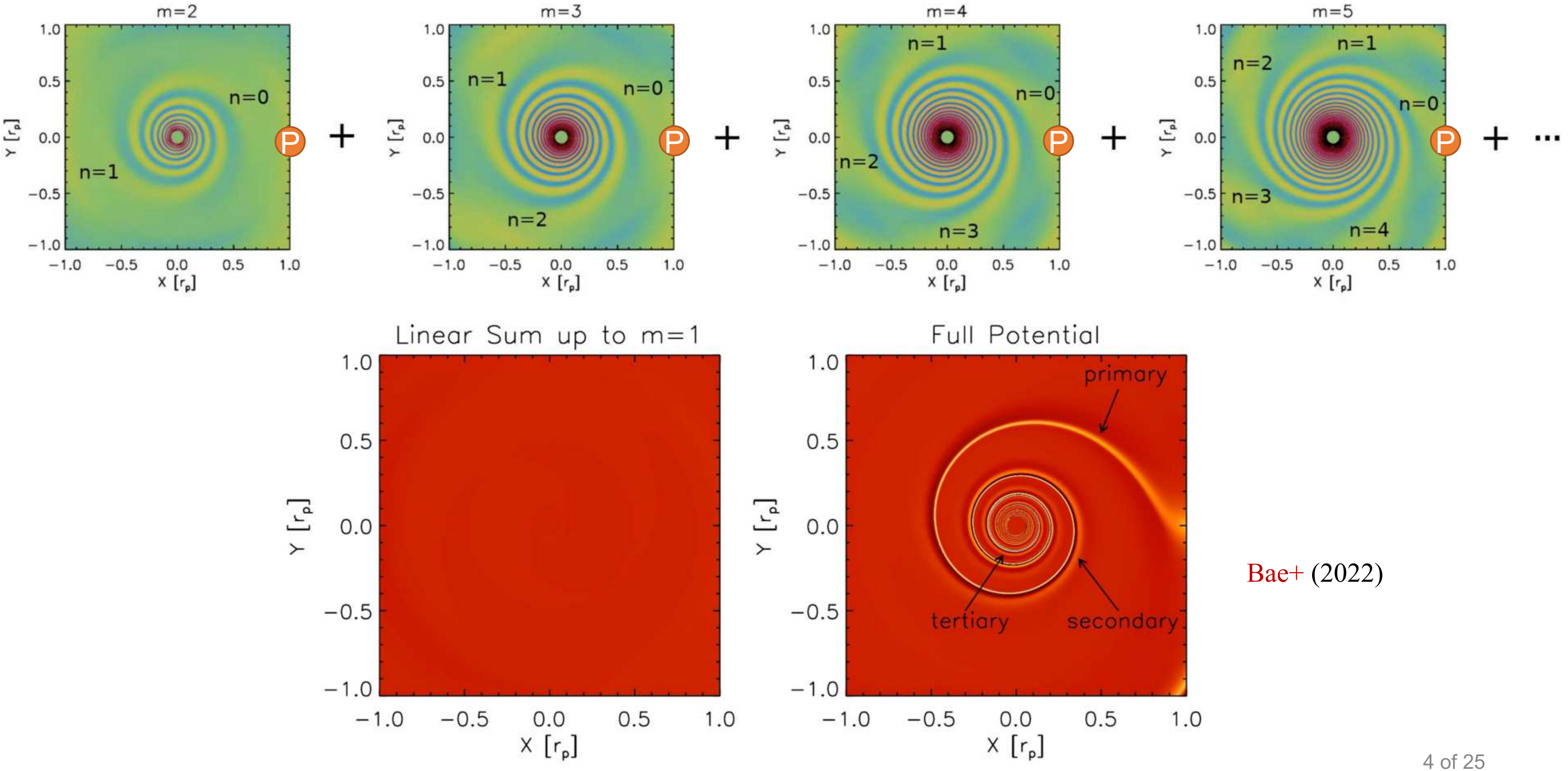
Simulation data from Verrios+ (2022)

Take Fourier modes (in azimuth) of planet potential:



$$\Phi_\mathrm{p}(r,\phi) = \sum_{m=0}^{\infty} \Phi_m(r,\phi) = \sum_{m=0}^{\infty} V_m(r)\cos(m\phi).$$

…and run a hydro simulation for each:



Bae+ (2022)

Spiral Wake Structure

How can we understand this?

Calculate the linear disk response

(+ WKBJ approximation)

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi G|k|\Sigma_0 + c_0^2 k^2$$

Lin & Shu (1964)

Lindblad resonances occur when $\Omega^2 = \mathrm{m}^2\left(\Omega - \Omega_p\right)^2$ since the solution explodes

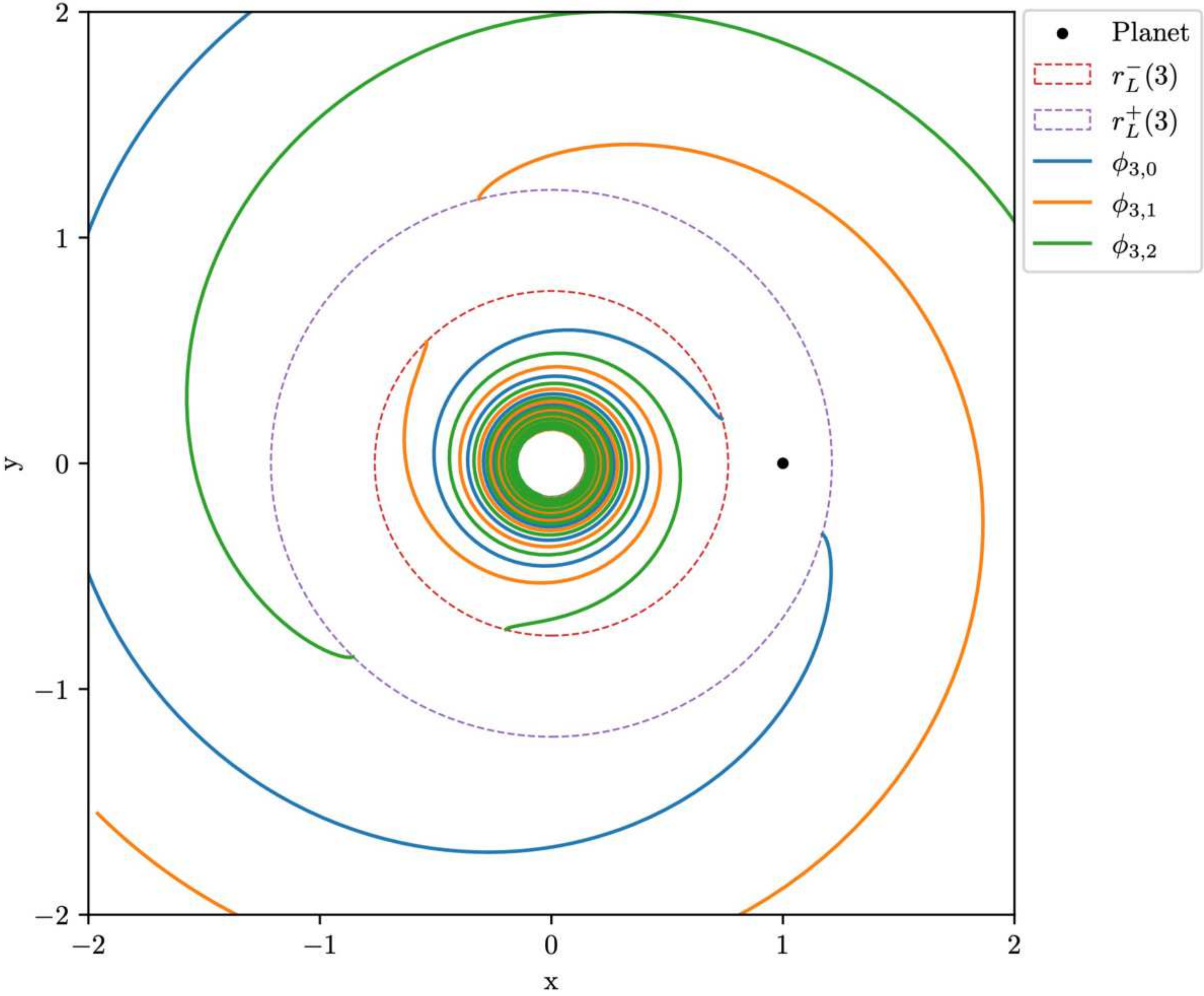
Spiral mode phase

Bae & Zhu (2018)

Obtain lines of constant phase

$$\phi_m(r) = \phi_m(r_{\rm L}^{\pm}) - \int_{r_{\rm L}^{\pm}}^{r} \frac{k(r')}{m}\, dr'$$

Pretty spirals!



$$m \to \infty$$

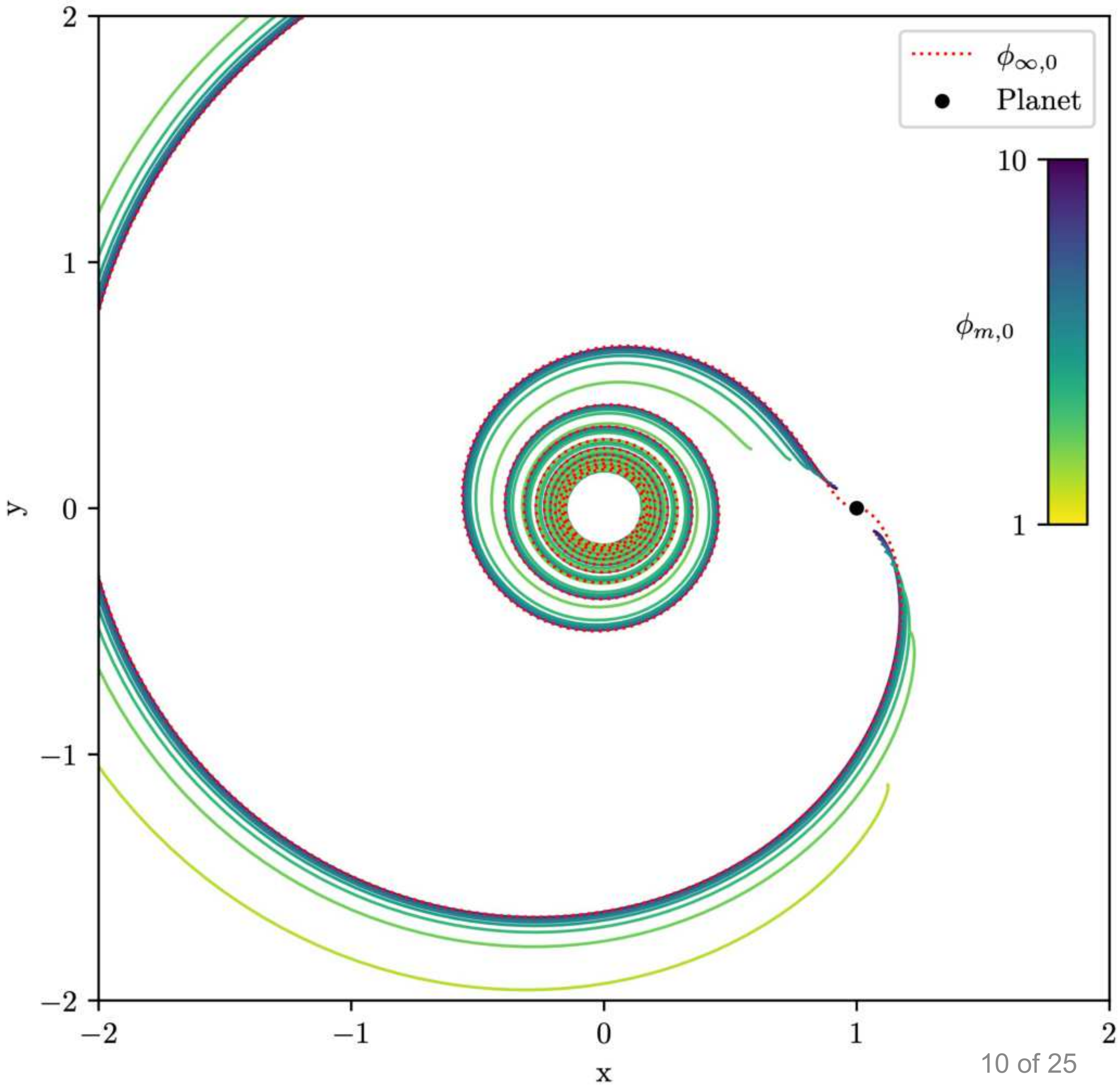
$$+$$

$$n = 0$$

$$=$$

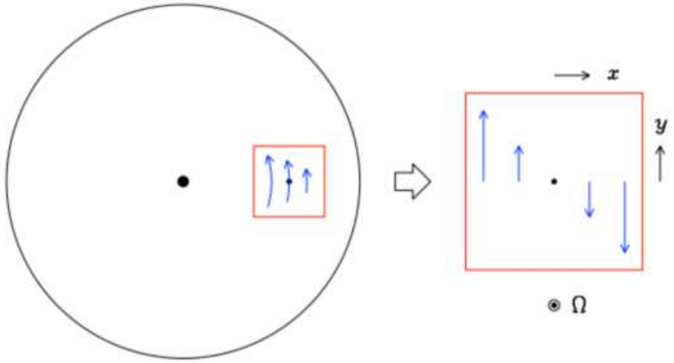
$$\phi_{\infty,0}(r) = -\int_{r_\mathrm{p}}^{r} \frac{\Omega(r') - \Omega_\mathrm{p}}{c(r')}\, dr'$$

Ogilvie & Lubow (2002)



Semi-Analytic Calculation

Local approximation



FFT

$$\frac{d^2\hat{v}}{d\tau^2} + \left[c^2k^2 + \kappa^2\right]\hat{v} = -ik_y\frac{d\hat{\Phi}_{\rm p}}{d\tau} + 2ik_xB\hat{\Phi}_{\rm p}$$

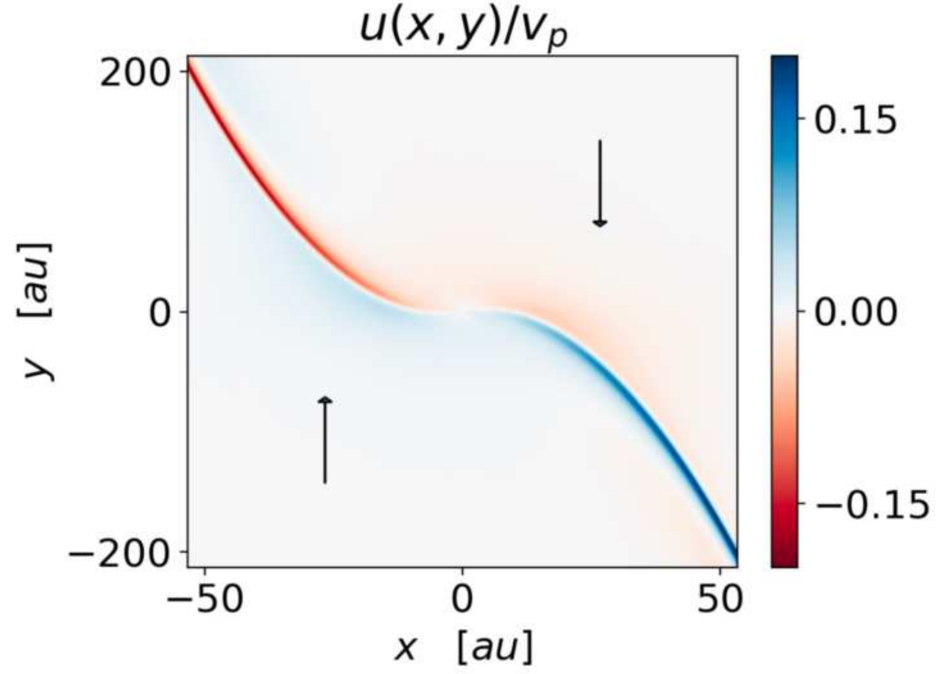
$$\hat{u} = -\frac{1}{c^2k_y^2 + 4B^2}\left(2B\frac{d\hat{v}}{d\tau} - c^2k_xk_y\hat{v} + 2iBk_y\hat{\Phi}_{\rm p}\right)$$

$$\hat{\sigma} = \frac{i}{c^2k_y^2 + 4B^2}\left(k_y\frac{d\hat{v}}{d\tau} + 2Bk_x\hat{v} + ik_y^2\hat{\Phi}_{\rm p}\right),$$

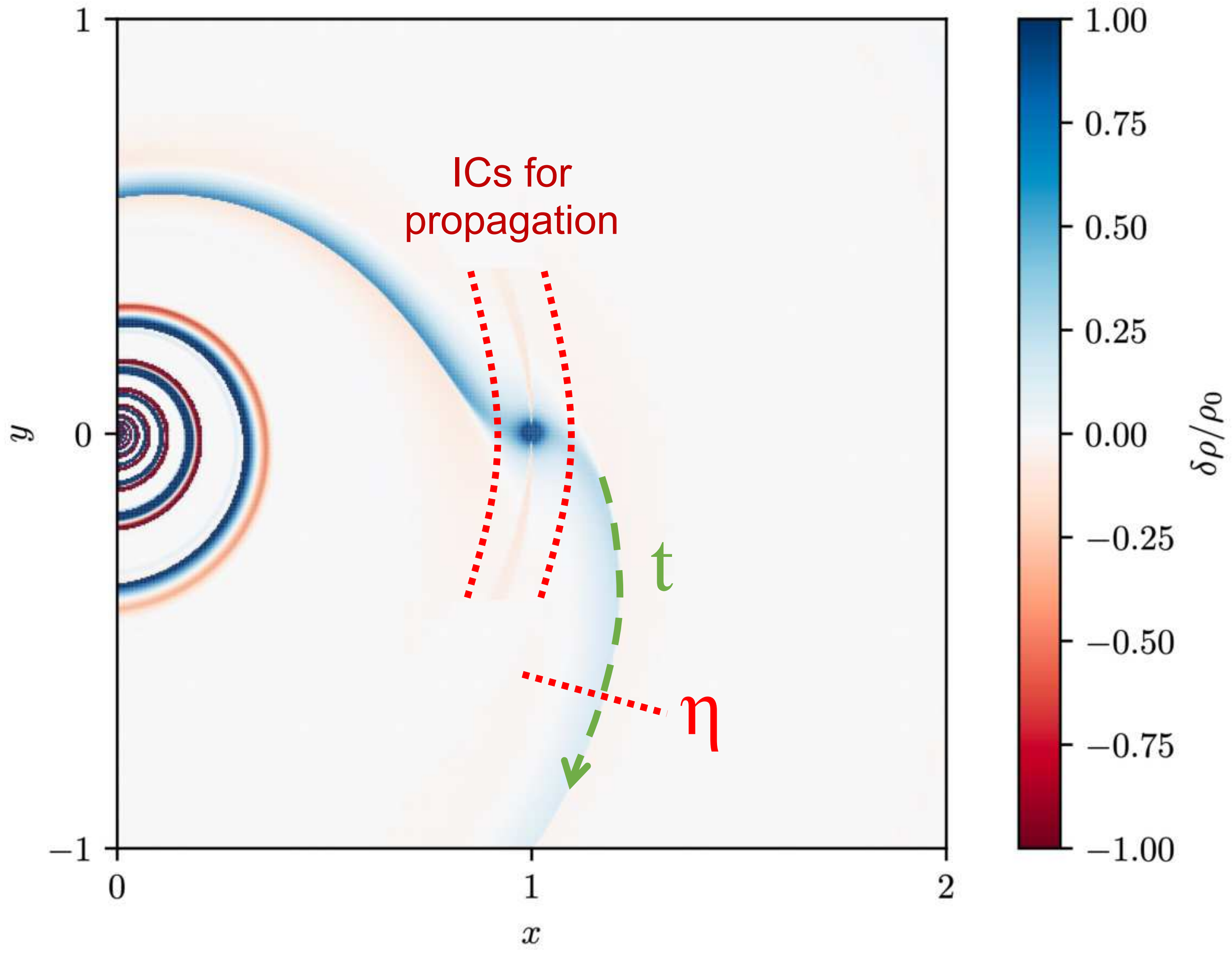
Goldreich & Tremaine (1978)

Linear wake generation

(solve nearby planet for Fourier components)



Bollati+ (2021)



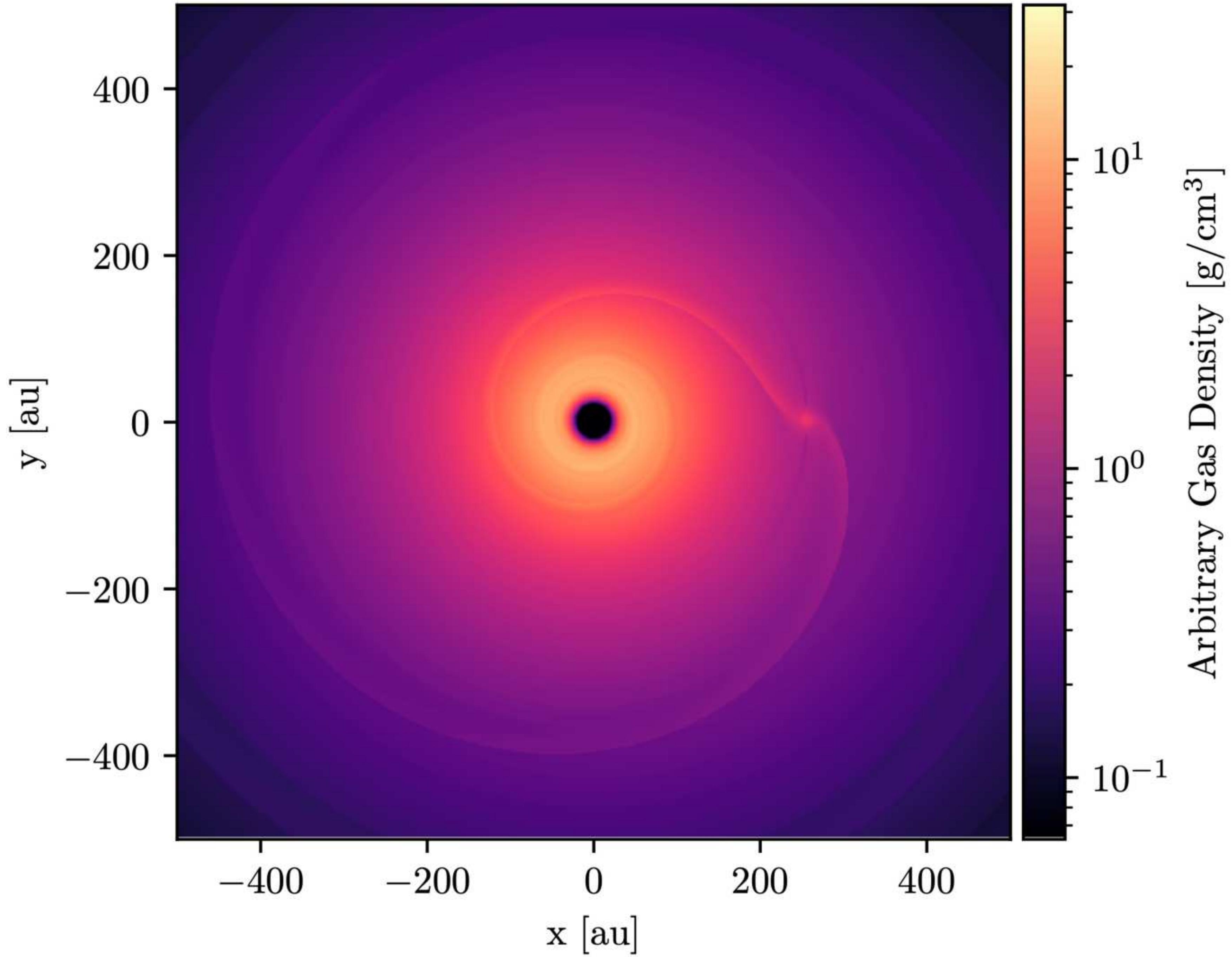
Non-linear wake propagation

(use linear as initial condition)

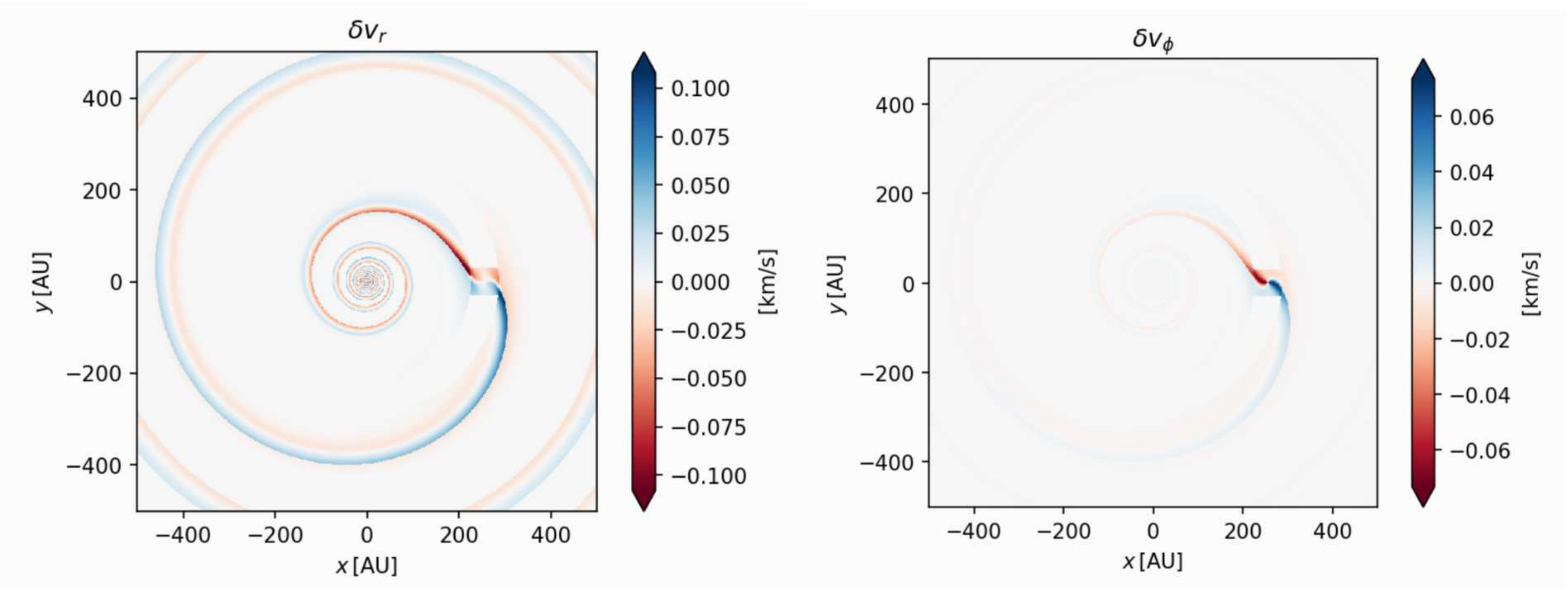
$$\partial_t \chi - \chi \partial_\eta \chi = 0$$

Solved with finite-volume (Godunov) scheme then mapped to real space

(If you are interested in the details, ask me afterwards! The derivation is very long)



400
200
0
−200
−400
y [au]
−400
−200
0
200
400
x [au]
10^1
10^0
10^{-1}
Arbitrary Gas Density [g/cm^3]


δv_r
400
200
0
−200
−400
y [AU]
−400
−200
0
200
400
x [AU]
0.100
0.075
0.050
0.025
0.000
−0.025
−0.050
−0.075
−0.100
[km/s]
δv_ϕ
400
200
0
−200
−400
y [AU]
−400
−200
0
200
400
x [AU]
0.06
0.04
0.02
0.00
−0.02
−0.04
−0.06
[km/s]

Wakeflow: A Python package for semi-analytic models of planetary wakes

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WAKEFLOW

Method improvements:

- Higher order accuracy for velocity components
- Adaptive time-stepping
- Efficiency (100x speed increase)
- More accurate initial condition retrieval
- \+ more

```
~ % pip install wakeflow
```

Velocity improvement

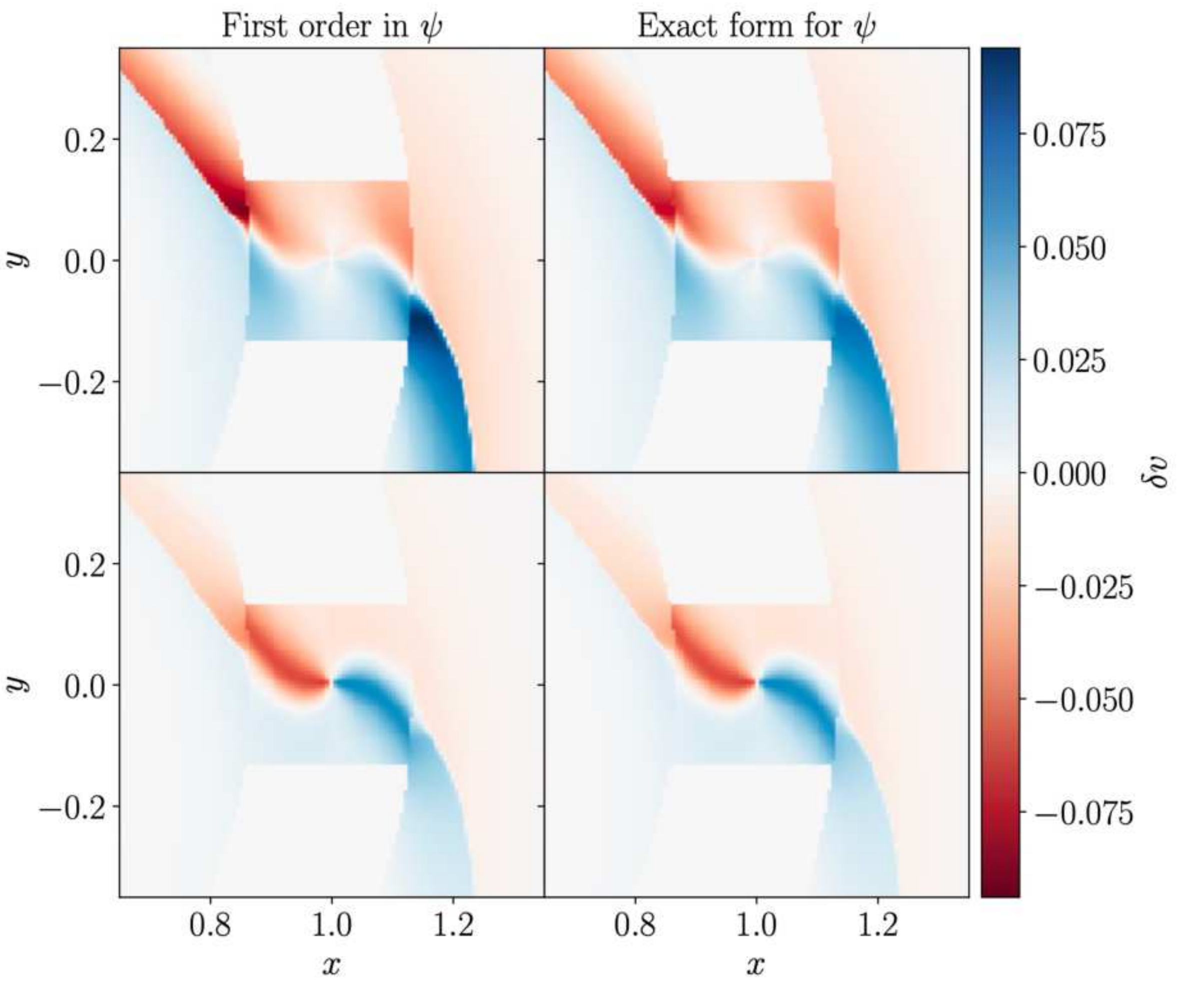
$$\psi = \frac{\gamma+1}{\gamma-1}\frac{c-c_0}{c_0}$$

$$c^2 = \frac{\partial P}{\partial \Sigma} = c_0^2(r)\left[\frac{\Sigma}{\Sigma_0(r)}\right]^{\gamma-1}$$

Rafikov (2002)

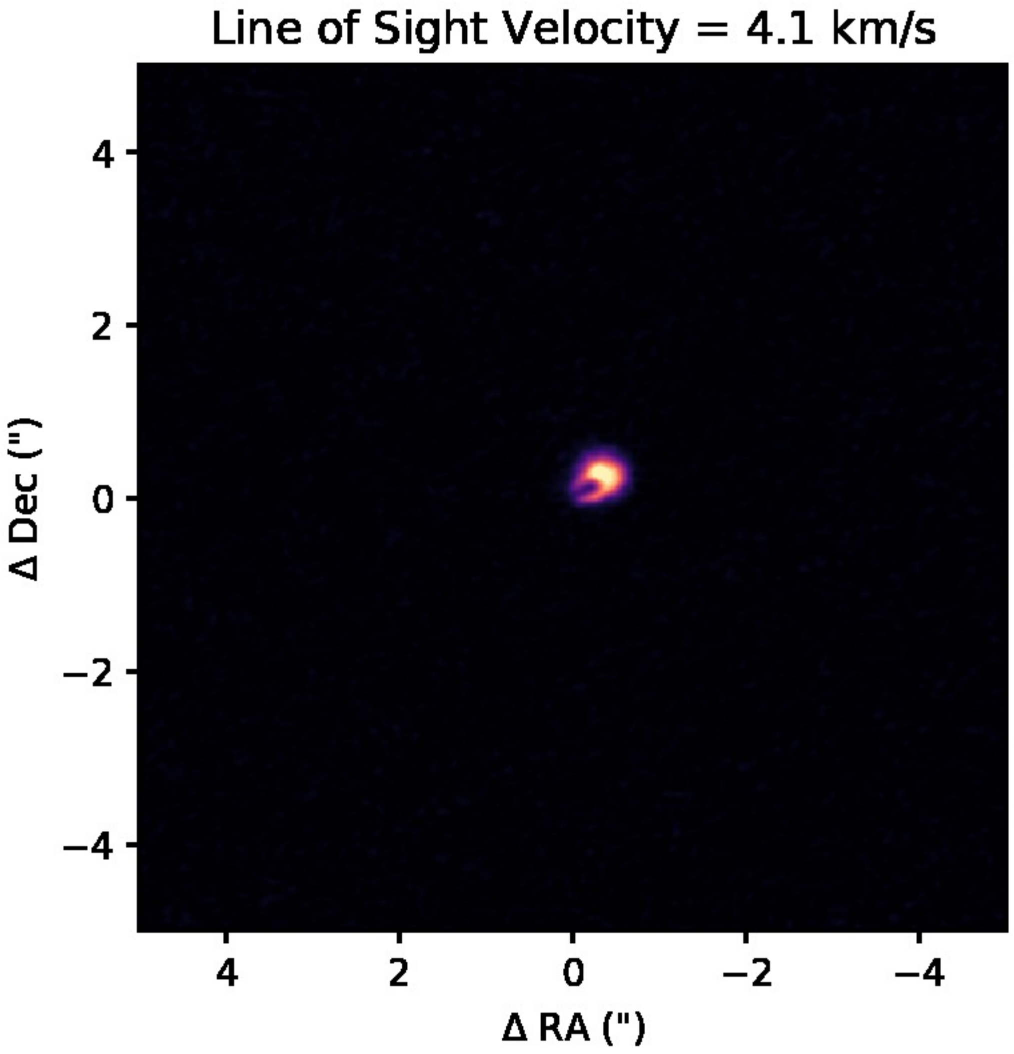
$$\frac{\Sigma-\Sigma_0}{\Sigma_0} = \frac{2}{\gamma+1}\psi + \frac{3-\gamma}{(\gamma+1)^2}\psi^2 + \mathcal{O}(\psi^3)$$

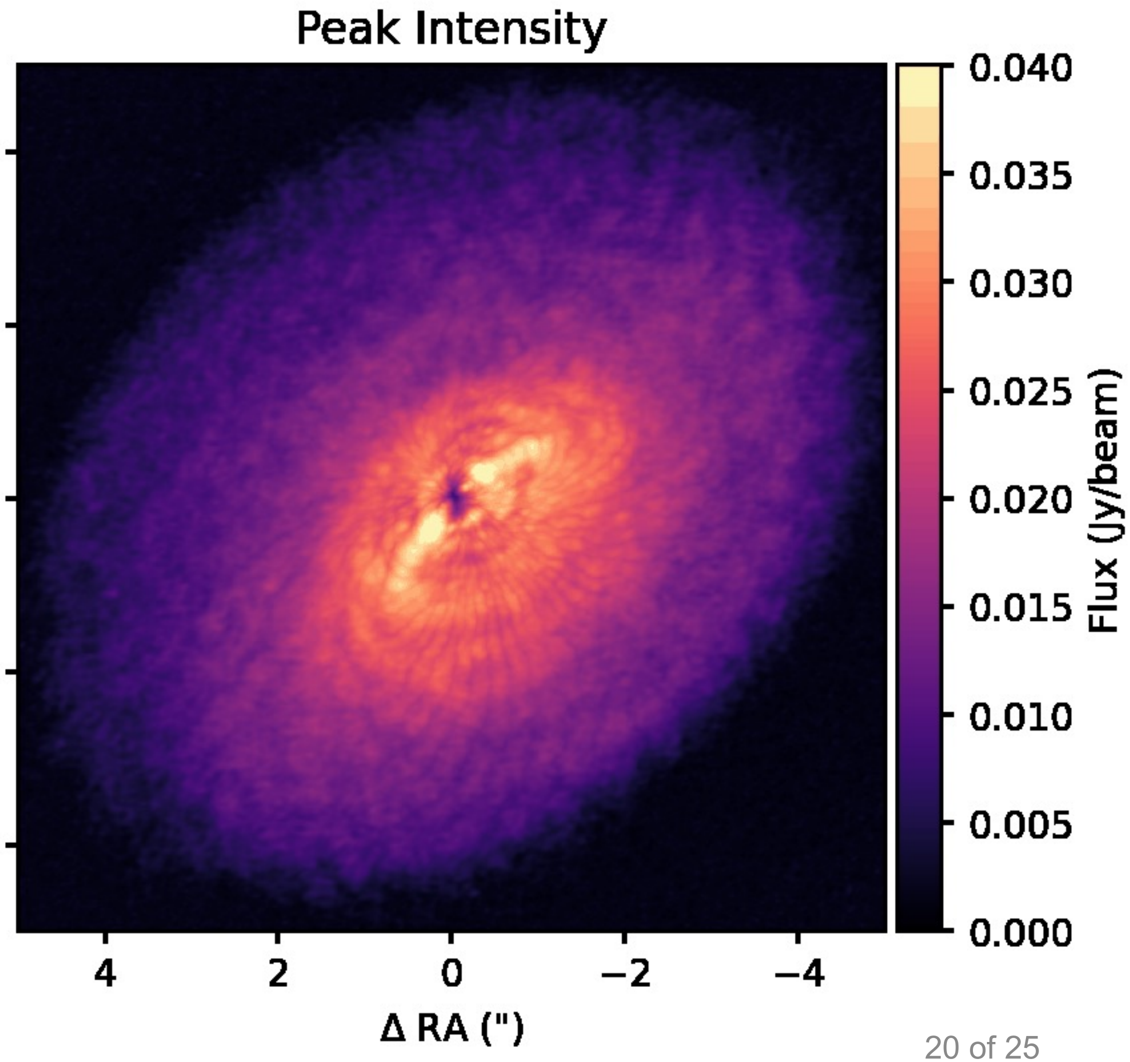
$$\psi = \frac{\gamma+1}{\gamma-1}\left[\left(\frac{\Sigma-\Sigma_0}{\Sigma_0}+1\right)^{(\gamma-1)/2}-1\right]$$



Applications

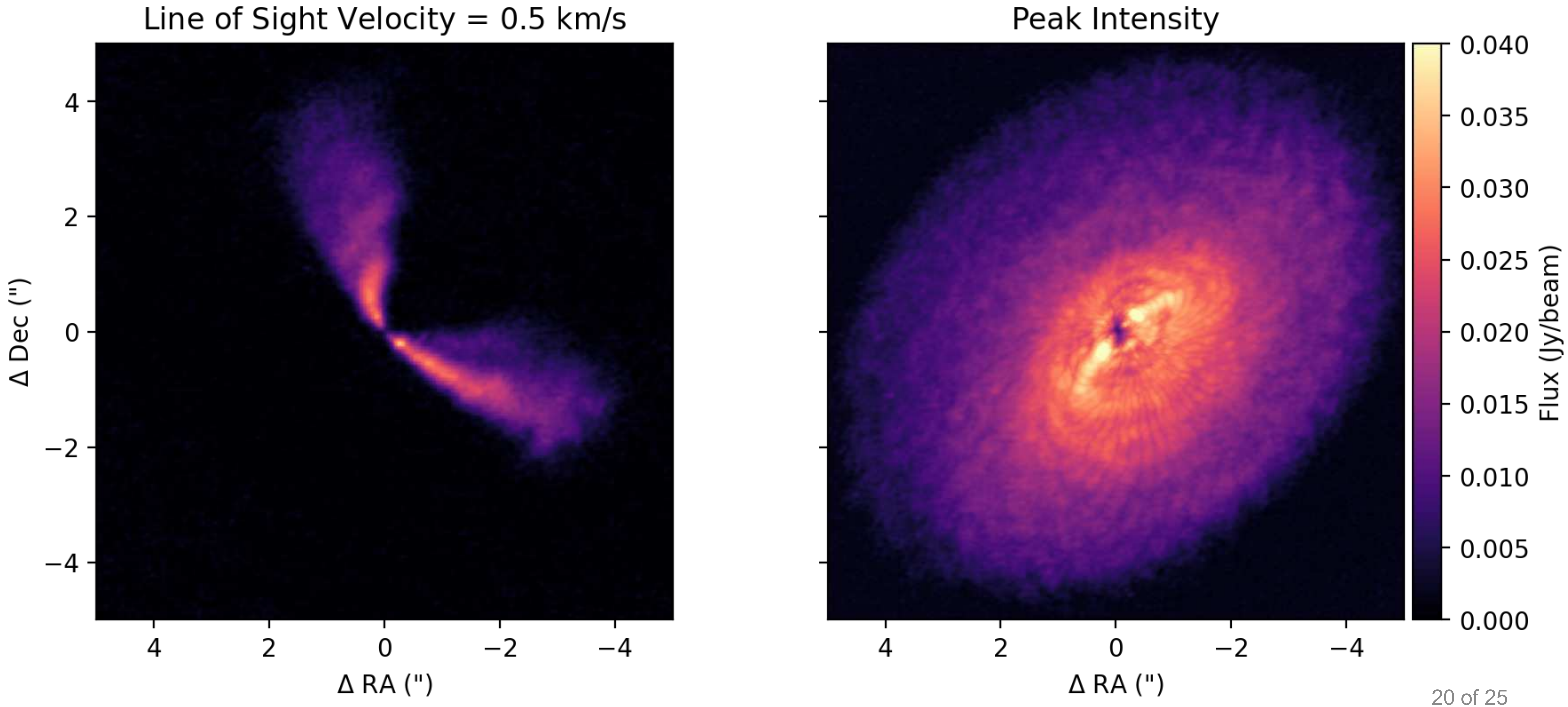
Disk Kinematics



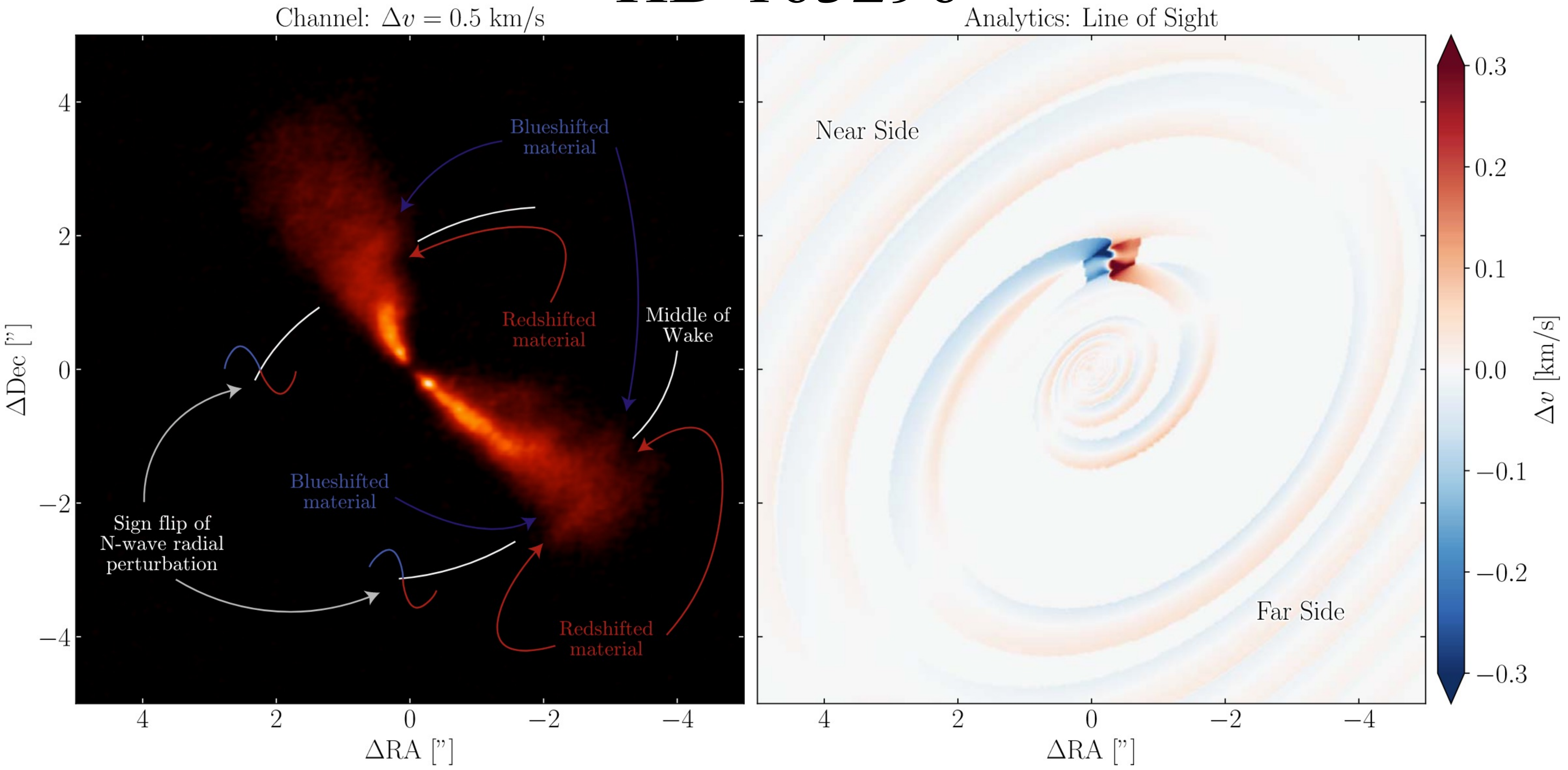


Data from Czekala et al. (2021); Oberg et al. (2021)

Disk Kinematics

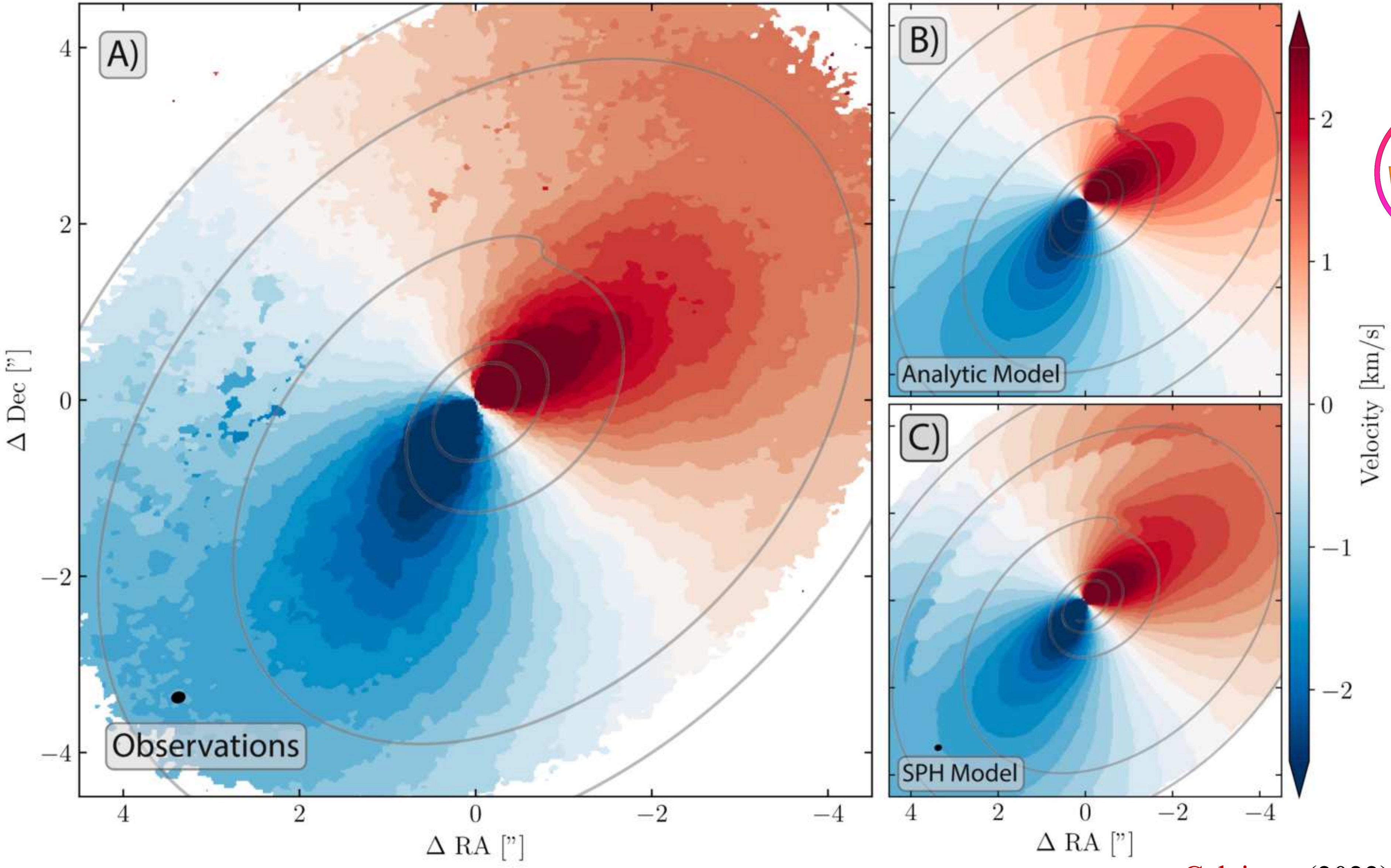


HD 163296



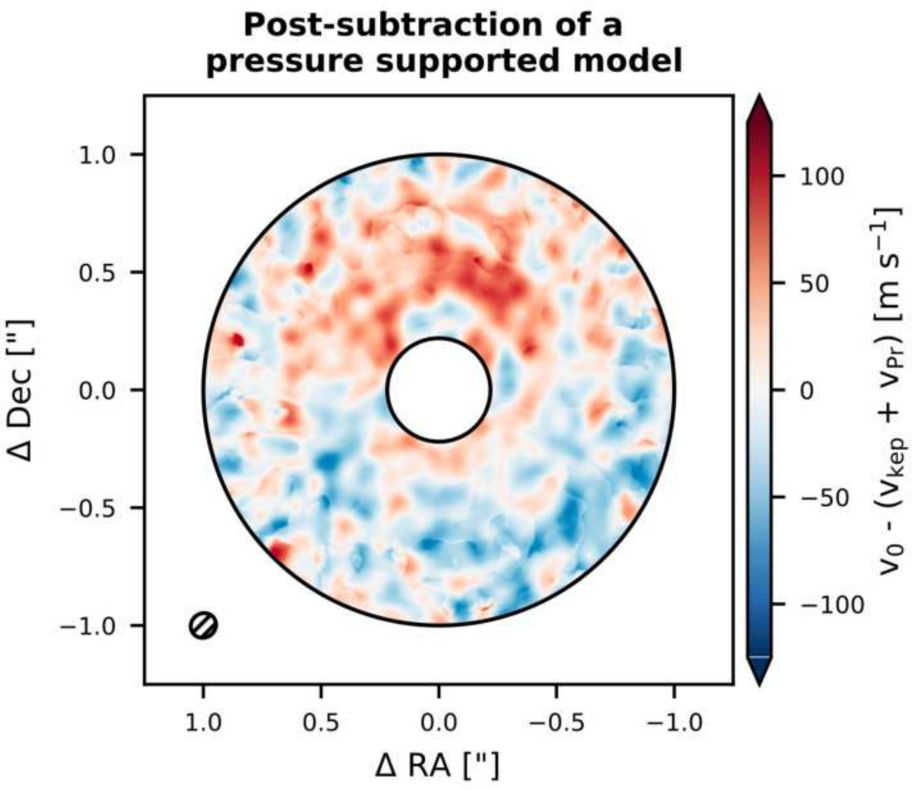
Calcino+ (2022)

HD 163296



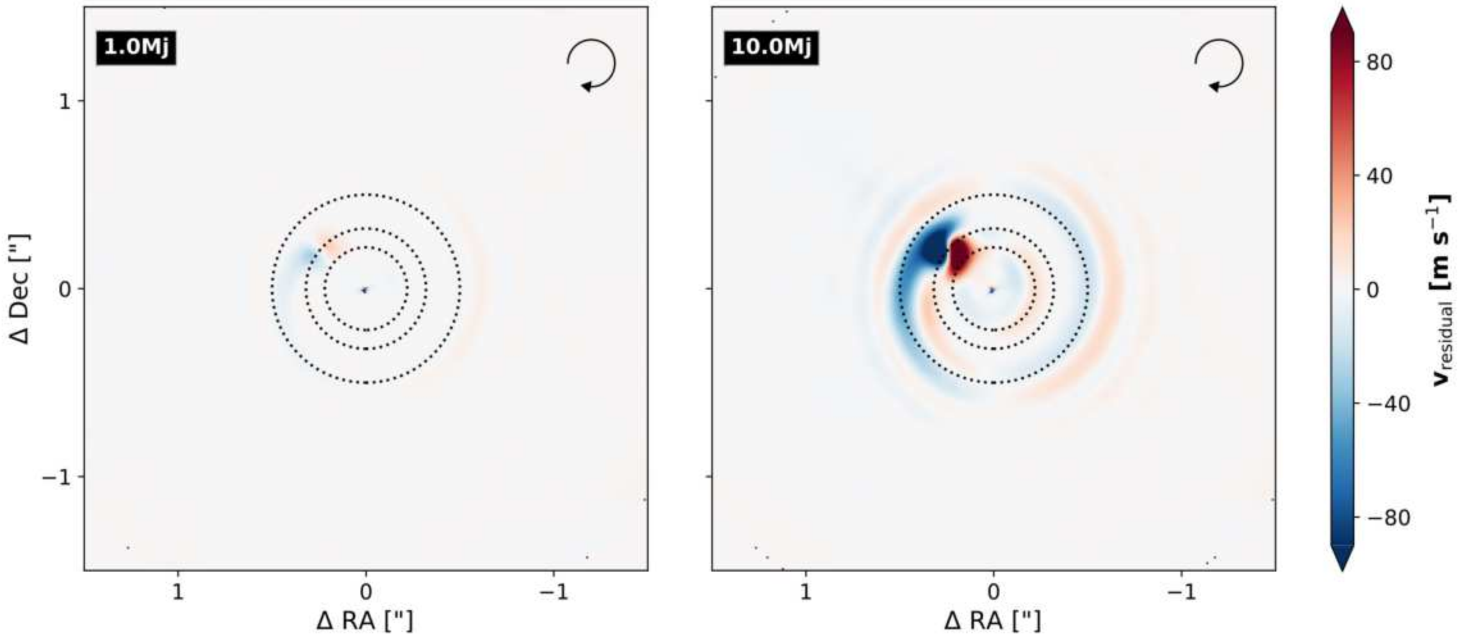
Calcino+ (2022)

HD 169142



WAKEFLOW + MCFOST

Garg+ (2022)



Other possible applications:

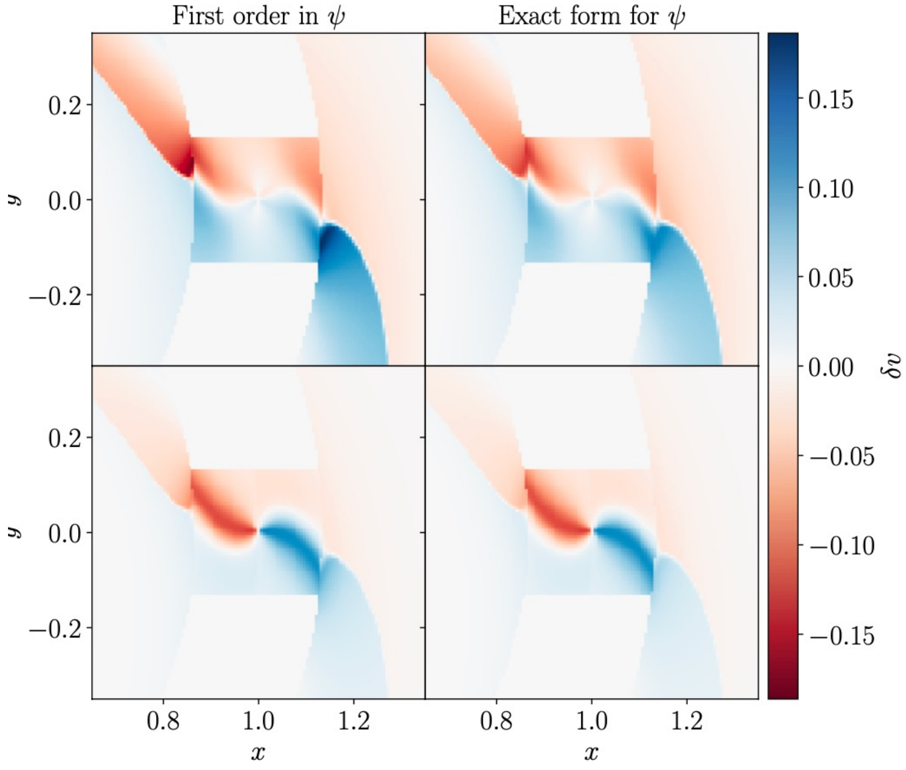
- Statistical tools:
  - Daniele Fasano working on incorporating Wakeflow into Discminer to measure planet masses from kinematics

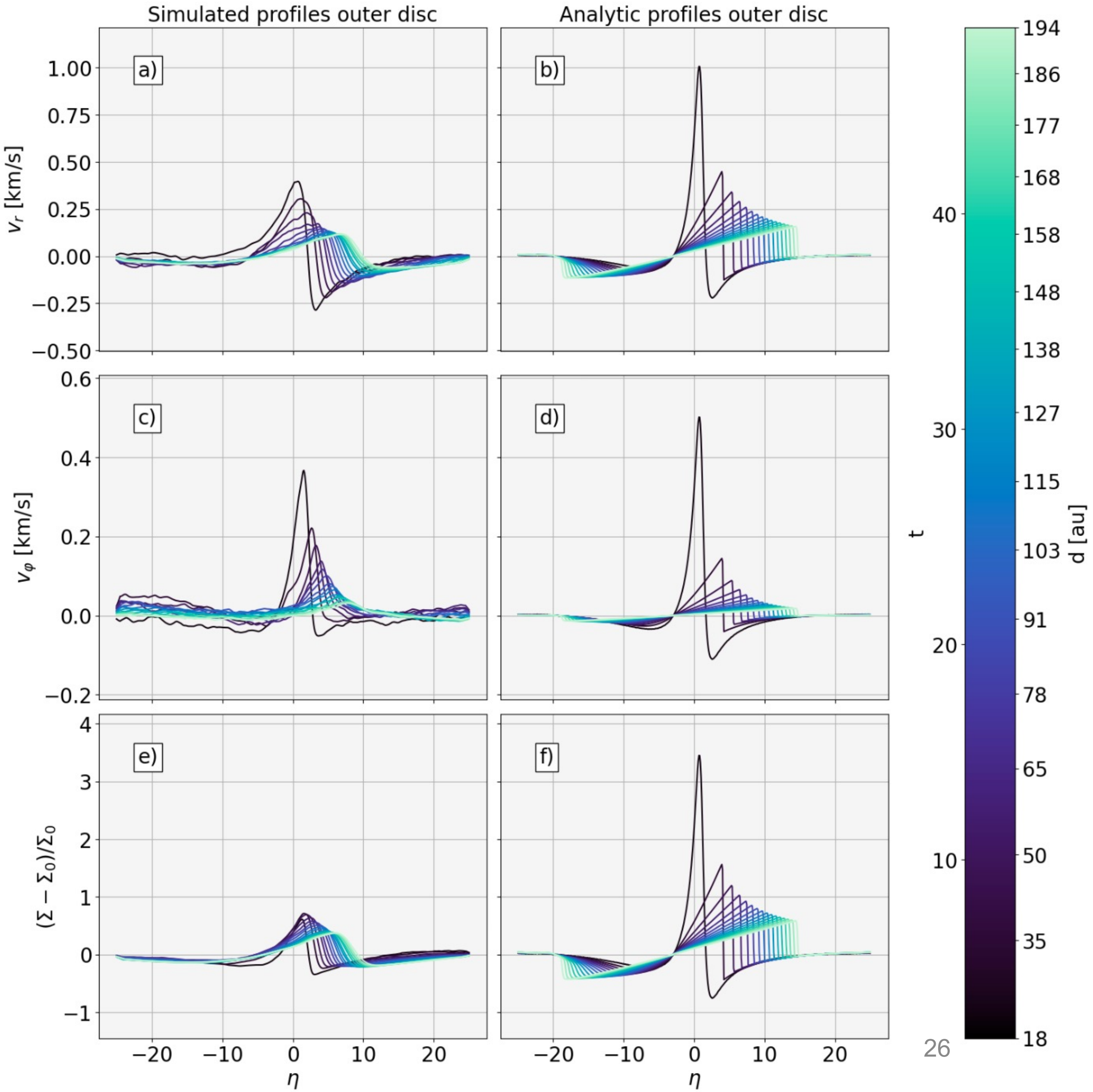
- Cheap parameter space exploration before expensive hydro models

+ your ideas?

Limitations

$$M_{\rm th} \equiv \frac{2}{3}\left(\frac{H_{\rm p}}{r_{\rm p}}\right)^3 M_\star$$





Temperature

We assume that the sound speed c_s obeys a simple radial power law:

$$c_s \propto R^{-q},$$

where R is a cylindrical radius coordinate and q is some real number. Thus temperature T must scale

$$T \propto c_s^2 \propto R^{-2q}.$$

Density

The density structure is derived asssuming the disk is in vertical hydrostatic equilibrium (eg. Pringle 1981). The density ρ is given by

$$\rho(R,z) = \rho(R_{\rm ref}) \left(\frac{R}{R_{\rm ref}}\right)^{-p} \exp\left(\frac{GM_*}{c_s^2}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{R}\right]\right),$$

where z is the height, $R_{\rm ref}$ is some reference radius, p is some real number, G is the gravitational constant and M_* is the mass of the central star.

`wakeflow` also supports the commonly used "exponentially-tapered" density structure, given by

$$\rho(R,z) = \rho(R_{\rm ref}) \left(\frac{R}{R_{\rm ref}}\right)^{-p} \exp\left(-\left[\frac{R}{R_{\rm c}}\right]^{2-p}\right) \exp\left(\frac{GM_*}{c_s^2}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{R}\right]\right),$$

where $R_{\rm c}$ is the "critical radius". This density structure is automatically used in `wakeflow` if you specify `r_c` to be non-zero.

Velocities

The velocities are derived assuming radial force balance (eg. Nelson et al. 2013). The radial and vertical motions are set to zero, while the rotation is given by

$$\Omega(R,z) = \Omega_{\rm K}\left[-(p+2q)\left(\frac{H}{R}\right)^2 + (1-2q) + \frac{2qR}{\sqrt{R^2+z^2}}\right]^{1/2},$$

where $\Omega_{\rm K} = \sqrt{\frac{GM_*}{R^3}}$ is Keplerian rotation and $H = c_s/\Omega_{\rm K}$ is the disk scale height.

Surface Density

Very commonly the density structure in disk models is specified in terms of the surface density Σ, defined by

$$\Sigma = \int_{-\infty}^{\infty} \rho \, dz.$$

Assuming we care only about regions of the disk where $z \ll R$, one can show that density and surface density are related by (see for example the lecture notes by Armitage, 2022)

$$\Sigma = \sqrt{2\pi} H \rho.$$

Thus if you parameterise the disk density by $\Sigma \propto R^{-\gamma}$, then γ is related to p and q by

$$p = \frac{3}{2} - q + \gamma.$$

Thus p and γ are not in general the same, although it is tempting to think that they could be.

Additionally the t transformation becomes (Rafikov, 2002a)

$$t(r) = \frac{3}{2^{5/4}} \left(\frac{r_\mathrm{p}}{H_\mathrm{p}}\right)^{\frac{5}{2}} \left| \int_1^{r/r_\mathrm{p}} |s^{\frac{3}{2}} - 1|^{\frac{3}{2}} s^{\frac{5q+\delta}{2} - \frac{11}{4}} \, ds \right|, \tag{4.10}$$

where explicitly the g function is given by (Bollati et al., 2021)

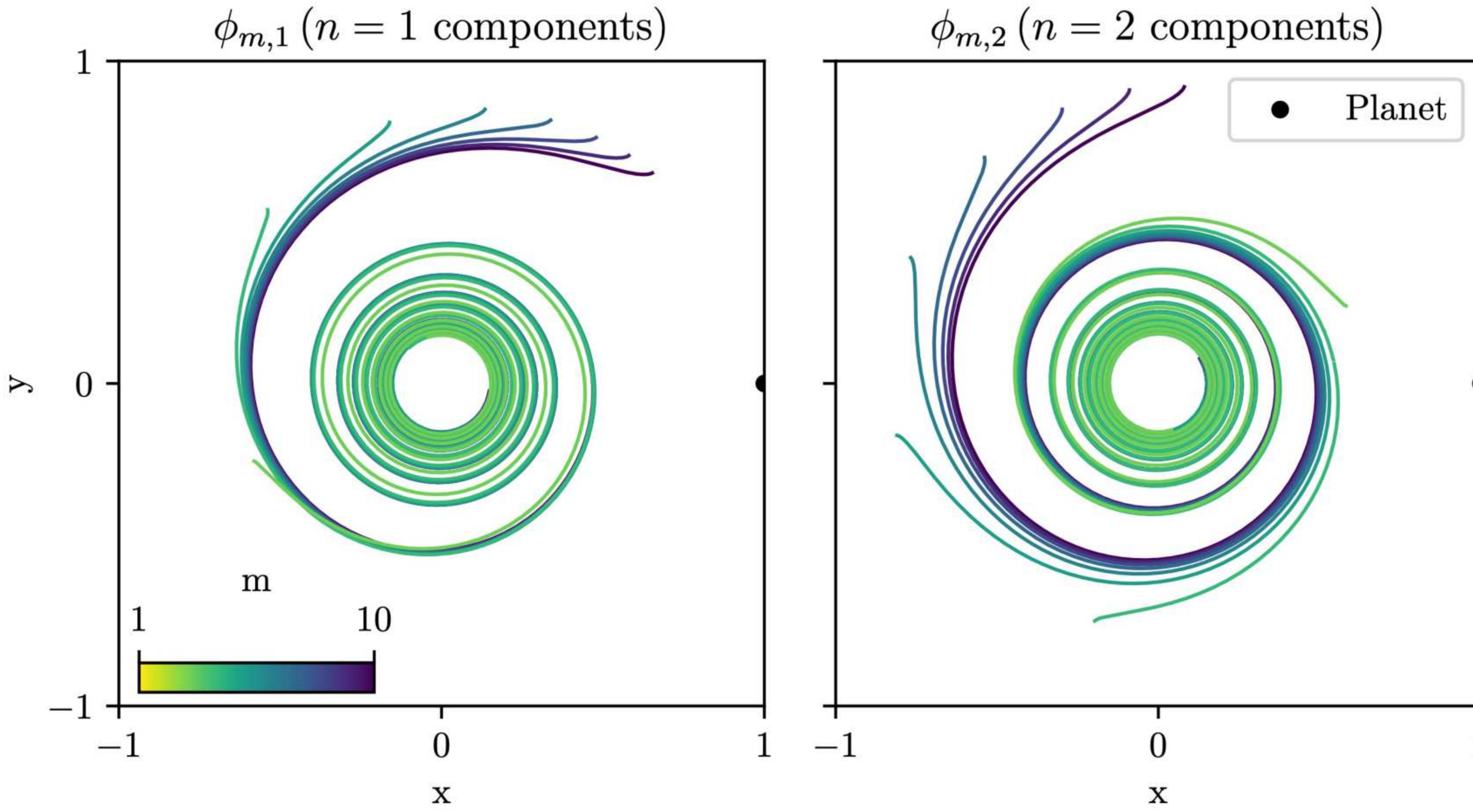
$$g(r) = 2^{1/4} \left(\frac{r_\mathrm{p}}{H_\mathrm{p}}\right)^{\frac{1}{2}} \frac{\left(\frac{r}{r_\mathrm{p}}\right)^{\frac{5}{4} - \frac{\delta+3q}{2}}}{\left|1 - \left(\frac{r}{r_\mathrm{p}}\right)^{\frac{3}{2}}\right|^{\frac{1}{2}}} \tag{4.11}$$

$$\frac{dt(s)}{ds} = \frac{r_\mathrm{p}}{l_\mathrm{p}} \left[\frac{\Omega(s) - \Omega_\mathrm{p}}{c_0(s) g(s)}\right]; \quad t(r_\mathrm{p}) = 0. \tag{4.12}$$

where obtaining $t(r)$ from the solution $t(s)$ is simply a matter of taking $s = r$. Applying this analysis to the t transformation for a power law disk (4.10) we obtain

$$\frac{dt(s)}{ds} = \frac{3}{2^{5/4}} \left(\frac{r_\mathrm{p}}{H_\mathrm{p}}\right)^{\frac{5}{2}} \left| |s^{\frac{3}{2}} - 1|^{\frac{3}{2}} s^{\frac{5q+\delta}{2} - \frac{11}{4}} \right|; \quad t(1) = 0, \tag{4.13}$$

$$s = r/r_\mathrm{p}.$$



$\phi_{m,1}$ ($n = 1$ components)
$\phi_{m,2}$ ($n = 2$ components)
Planet
y
x
x
m
1
10
1
0
−1
−1
0
1
−1
0
1